PME 42

July 3-8, 2018 --- Umeå, Sweden

Proceedings

Of the 42nd Conference of the International Group
for the Psychology of Mathematics Education

Editors: Ewa Bergqvist, Magnus Österholm,
Carina Granberg, and Lovisa Sumpter

Volume 1

**Plenary Lectures, Plenary Panel, Research Forums,
Working Groups, Seminars, Colloquia, National Presentation**

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PREFACE

We are delighted to welcome you to PME 42, the 42nd Annual Conference of the International Group for the Psychology of Mathematics Education. PME is one of the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world. This year we had submissions from 674 persons from 53 different countries, and around 650 people are expected to attend the conference.

PME 42 convenes in Umeå, in the north of Sweden. This is the first time PME is organized in Sweden, the second time in Scandinavia (PME 28 was in Bergen, Norway), and the third time in the Nordic countries (PME 21 was in Lahti, Finland). Umeå is located 64 degrees north, which is the furthest north PME has ever taken place. Participants will be able to experience the truly magical summertime in the north of Sweden. The lovely bright summer nights, when the sun barely sets, and it is just as light in the late evening as in the middle of the day, make a truly spectacular experience that no visitor will ever forget.

Delight in Mathematics Education is the theme of the conference, which refers to the joy, pleasure, and beauty in both mathematics and mathematics education. It includes issues on how both teaching and learning mathematics can be fun, meaningful, and inspiring, for teachers as well as for students. The theme also encompasses how mathematics and mathematics education can connect to individual students and teachers, for example, through aspects of motivation, creativity, and usefulness, and how individuals can see themselves as able in mathematics. The *light* in the conference theme also alludes to the bright summer nights in Umeå.

PME 42 includes two types of sessions that exist for the first time in PME. First, the previous group formats Working Session and Discussion Group have this year been combined to a new group format named Working Group. Second, it has been possible to submit Colloquia proposals to PME for a few years, but until this year, no Colloquium has ever been included in the program. PME 42 includes two Colloquia.

The papers in the five volumes of these proceedings are organized according to type of presentation. Volume 1 contains the Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminars, Colloquia, and the National Presentation of mathematics education in Sweden. Volumes 2–4 contain the Research Reports, while Volume 5 consists of the Oral Communications and Poster Presentations.

The organization of PME 42 is a collaborative effort involving colleagues in Umeå Mathematics Education Research Centre (UMERC), at Umeå University, but also colleagues from other Swedish universities. The conference is organized with the support of three committees: the International Program Committee for PME 42, the International Committee of PME together with the PME Administrative Manager, and the Local Organizing Committee. We acknowledge the support and effort of all involved in making the conference possible and thank each and every one of you. Finally, we thank each PME participant for making your journey to PME 42 in Umeå and for your contributions to this conference.

Our goal is to make PME 42 scientifically and socially successful. We hope you find your participation fruitful and memorable—and that you experience *delight* during the conference and during your visit in Umeå.

Ewa Bergqvist and Magnus Österholm

PME 42 Conference Chairs

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TABLE OF CONTENTS

VOLUME 1

PREFACE ... iii
TABLE OF CONTENTS VOLUME 1 ... v
TABLE OF CONTENTS VOLUME 2 ... ix
TABLE OF CONTENTS VOLUME 3 ... xv
TABLE OF CONTENTS VOLUME 4 ... xxi
TABLE OF CONTENTS VOLUME 5 ... xxvii
THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME) ... liii
PROCEEDINGS OF PREVIOUS PME CONFERENCES ... lvii
THE PME 42 CONFERENCE ... lxi
REVIEW PROCESS OF PME 42 ... lxiii
LIST OF PME 42 REVIEWERS ... lxv

PLENARY LECTURES

SYSTEMS PERSPECTIVES ON MATHEMATICS TEACHERS' BELIEFS: ILLUSTRATIONS FROM BELIEFS ABOUT STUDENTS ... 3
Kim Beswick
FROM ANXIETY TO ENGAGEMENT: HISTORY AND FUTURE OF RESEARCH ON MATHEMATICS-RELATED AFFECT ... 19
Markku Hannula
THE VERY MULTI-FACETED NATURE OF MATHEMATICS EDUCATION RESEARCH ... 35
Mogens Niss
AN AESTHETIC TURN IN MATHEMATICS EDUCATION ... 51
Nathalie Sinclair

PLENARY PANEL

CHICKEN-EGG CYCLES: WHAT NEEDS TO COME FIRST, HIGH PERFORMANCE OR POSITIVE AFFECTIVE VARIABLES REGARDING MATHEMATICS? ... 69
Márcia Maria Fusaro Pinto

WHY MOTIVATION NECESSARILY PRECEDES HIGH MATHEMATICAL PERFORMANCE – BUT NOT VICE VERSA 77
Wim Van Dooren, Qiaoping Zhang

POSITIVE AFFECTIVE VARIABLES REGARDING MATHEMATICS FOLLOW HIGH PERFORMANCE 85
Francesca Morselli, Wee Tiong Seah

RESEARCH FORUMS

OTHERNESS IN MATHEMATICS EDUCATION 95
Yasmine Abtahi, Jill Adler, David Guillemette, Rune Herheim, Steve Lerman, Jean-François Maheux, Paola Valero

LEARNING AND TEACHING OF ARITHMETIC SKILLS IN EARLY YEARS 125
Camilla Björklund, Angelika Kullberg, Ulla Runesson Kempe, Maria Reis, Douglas Clements, Julie Sarama, Esther Levenson, Ruthi Barkai, Dina Tirosh, Pessia Tsamir, Michael Askew, Hamsa Venkat, Minna Hannula-Sormunen

VITAL SIGNS OF COLLECTIVE LIFE IN THE CLASSROOM 155
Lynn M. McGarvey, Jennifer S. Thom, Joyce Mgombelo, Jérôme Proulx, Elaine Simmt, Florence Glanfield, Brent Davis, Lyndon Martin, Jo Towers, Charlotte Bertin, Kevin Champagne, Rox-Anne L'Italien-Bruneau, Charlotte Mégrourèche

WORKING GROUPS

INTEGRATING MATHEMATICS IN STEM EDUCATION: AN INTERNATIONAL PERSPECTIVE 187
Judy Anne Anderson, Yeping Li

TEACHER TENSIONS AS A LENS TO UNDERSTAND TEACHERS' RESISTANCE TO CHANGE 189
Chiara Andrà, Pietro Di Martino, Peter Liljedahl, Annette Rouleau

ARGUMENTATION AND FORMATIVE ASSESSMENT: A POSSIBLE SYNERGY? 191
Annalisa Cusi, Francesca Morselli, Cristina Sabena

INSTRUMENTS AND THE BODY 193
Paul Drijvers, Francesca Ferrara

REPLICATION IN MATHEMATICS EDUCATION 195
Matthew Inglis, Stanislaw Schukajlow, Wim Van Dooren, Markku Hannula

INTERNATIONAL PERSPECTIVES: MEASURING MATHEMATICS TEACHERS' KNOWLEDGE IN THE DIGITAL ERA 197
Guo Kan, Hege Kaarstein, Agida Manizade, Chandra Orrill

EXPLORING THE ROLE OF FACILITATORS IN VIDEO-BASED PROFESSIONAL DEVELOPMENT FOR MATHEMATICS TEACHERS 199
Ronnie Karsenty, Alf Coles

LEARNING MATHEMATICS IN/THROUGH/BY ARTS PRACTICES 201
Ricardo Nemirovsky, Kate O'Brien, Nathalie Sinclair

MATHEMATICS EDUCATION RESEARCH FROM AND IN LATIN AMERICA 203
Vanessa Neto, Raimundo Elicer, Gustavo Bruno

ASSESSMENT FOR LEARNING IN DIVERSE CLASSROOMS 205
Guri Nortvedt, Leonor Santos, Anna Teledahl

MATHEMATICAL LEARNING DISABILITIES: A CHALLENGE FOR MATHEMATICS EDUCATION 207
Cecile Ouvrier-Buffet, Elisabetta Robotti, Thierry Dias, Marie-Line Gardes

EYE-TRACKING IN MATHEMATICS EDUCATION RESEARCH: A FOLLOW-UP ON OPPORTUNITIES AND CHALLENGES 209
Maike Schindler, Eeva Haataja, Achim J. Lilienthal, Enrique Garcia Moreno-Esteva, Anna Shvarts

CONSIDERING THE DESIRED TEACHER OF MATHEMATICS TEACHER EDUCATION 211
Kicki Skog, Iben Maj Christiansen, Lisa Österling

WORKING GROUP: THE DESIGN OF INTENDED MATHEMATICS CURRICULA 213
Andreas J. Stylianides, William G. McCallum, Lynne McClure, Aisling Twohill

MATHS AND SPECIAL EDUCATION WORKING GROUP 215
Helen Rachel Thouless, Robyn Ruttenberg-Rozen, Ron Tzur

SEMINAR

REVIEWING FOR PME - A PRIMER FOR (NEW) REVIEWERS 219
David Maximiliano Gomez, Anika Dreher

COLLOQUIUM 1

PARALLEL ANALYSES OF COLLABORATIVE MATHEMATICS PROBLEM SOLVING IN A LABORATORY CLASSROOM SETTING 223
Man Ching Esther Chan (organizer), Markku Hannula (discussant)

ENTANGLED MODES OF SOCIAL INTERACTION IN STUDENT COLLABORATIVE PROBLEM SOLVING IN MATHEMATICS: CONNECTING PROCESS AND PRODUCT 225
Man Ching Esther Chan, May Ee Vivien Wan, David Clarke

ANALYSING PATTERNS OF STUDENTS' INTERACTION WHEN SOLVING OPEN-ENDED TASKS IN SMALL GROUPS 233
Javier Diez-Palomar

ANALYZING ENGAGEMENT IN MATHEMATICAL COLLABORATION: WHAT CAN WE SAY WITH CONFIDENCE? 235
Laura Tuohilampi

COLLOQUIUM 2

FOREGROUNDING DAVYVOD'S CURRICULUM: RELATIONAL APPROACH AND ALGEBRAIC THINKING IN EARLY GRADES 245
Elena Polotskaia (organizer), Cristina Sabena (discussant)

ALGEBRAIC THINKING AMONG GRAPHICAL REPRESENTATIONS AND ALGEBRAIC SYMBOLS 247
Maria Mellone, Gemma Carotenuto, Rosa Di Bernardo, Colomba Punzo

IDENTIFYING ALGEBRAIC REASONING ABOUT FRACTIONS 255
Helena Eriksson

MULTIPLICATIVE STRUCTURES IN ELEMENTARY SCHOOL MATHEMATICS: RELATIONAL APPROACH 263
Elena Polotskaia, Annie Savard, Marie-Sophie Gélinas, Osnat Fellus

NATIONAL PRESENTATION

MATHEMATICS EDUCATION RESEARCH IN SWEDEN: NATIONAL PRESENTATION AT PME 42 273
Ola Helenius, Cecilia Kilhamn, Peter Nyström

INDEX

INDEX OF AUTHORS AND CO-AUTHORS (ALL VOLUMES) 301

TABLE OF CONTENTS

VOLUME 2

RESEARCH REPORTS A-HAA

WHY IS CALCULATING THE AVERAGE SPEED DIFFICULT? 3
Linda Marie Ahl, Ola Helenius

ACCOUNTABILITY AND ASSESSMENT: GAPS AND GRIDS 11
Julie Alderton, Nick Pratt

PRESERVICE MATHEAMTICS TEACHERS' CURRICLUM VISUALIZATION 19
Julie Amador, Darrell Earnest

LINKS BETWEEN TEACHERS' PEDAGOGICAL TECHNOLOGICAL KNOWLEDGE AND THEIR PERSONAL CHARACTERISTICS 27
Ahlam Adnan Anabousy, Michal Tabach

MAKING MATHS MATTER: ENGAGING STUDENTS FROM LSES SCHOOLS THROUGH SOCIAL JUSTICE CONTEXTS 35
Judy Anne Anderson, Alice Kriesler

CLASSROOM DIALOGUE AS A FRENCH BRAID: A CASE STUDY FROM TRIGONOMETRY 43
Mette Andresen, Bettina Dahl

USING SELF-VIDEO ANALYSIS TO PROMOTE TEACHER GROWTH 51
Nick Andrews, Jenni Ingram, Andrea Pitt

DEVELOPING PROSPECTIVE TEACHERS' MATHEMATICS ORIENTATIONS IN THE CONTENT COURSES 59
Aina Appova

BIG BLOCKS OF PROOF 67
Amir Hossein Asghari, Sharareh Taghi Dastjerdi, Fatemeh Ahmadpour Mobarakeh

MIDDLE GRADE STUDENTS' PERFORMANCE ON ARITHMETIC CALCULATIONS PRESENTED AS WORD PROBLEMS OR NUMERIC PROBLEMS 75
Mike Askew, Hamsa Venkat

MOTIVATION AND CANING IN GHANAIAN SECONDARY SCHOOL: EVIDENCE FROM A SURVEY AND INTERVIEWS 83
Florence Christianah Awoniyi

LEARNING TO ASSESS: EXPLORING CHANGES TO PRE-SERVICE TEACHERS' CRITERIA FOR A QUADRATICS TASK 91
Michal Ayalon, Karina Wilkie

ELEMENTARY SCHOOL TEACHERS' IMPLEMENTATION OF DYNAMIC GEOMETRY USING MODEL LESSON VIDEOS 99
Anna E. Baccaglini-Frank, Pietro Di Martino, Nathalie Sinclair

AFFORDANCES AND TENSIONS IN TEACHING BOTH COMPUTATIONAL THINKING AND MATHEMATICS.................................... 107
Sandy Bakos, Mathieu Thibault

THE TRANSITION FROM HIGH SCHOOL TO UNIVERSITY MATHEMATICS: ENTERING A NEW COMMUNITY OF PRACTICE............ 115
Amalia-Christina Bampili, Charalampos Sakonidis, Theodossios Zachariades

SITUATIONAL AND DISTAL SOURCES OF MEANING IN A MULTILINGUAL MATHEMATICS CLASSROOM.. 123
Richard Barwell

KNOWLEDGE OF STATISTICAL TESTS BY PROSPECTIVE HIGH SCHOOL TEACHERS 131
Carmen Batanero, María del Mar López-Martín, María Magdalena Gea, Pedro Arteaga

THE EFFECTIVENESS OF INTEGRATING GEOGEBRA HOTS ACTIVITIES ON THE DEVELOPMENT OF CREATIVE MATHEMATICAL THINKING 139
Nimer Fayez Baya'a, Wajeeh Mahmood Daher, Dania Jamal Abo-Mokh, Ahlam Adnan Anabousy

INDIVIDUAL DIFFERENCES IN FRACTIONS' CONCEPTUAL AND PROCEDURAL KNOWLEDGE: WHAT ABOUT OLDER STUDENTS?........... 147
Maria Bempeni, Stavroula Poulopoulou, Ioanna Tsiplaki, Xenia Vamvakoussi

PRIMARY SCHOOL CHILDREN'S (9 YEAR OLDS') UNDERSTANDING OF QUADRILATERALS 155
Melania Bernabeu, Mar Moreno, Salvador Llinares

VALUE OF PICTURES IN MODELLING PROBLEMS FROM THE STUDENTS' PERSPECTIVE 163
Matthias Böckmann, Stanislaw Schukajlow

CALCULUS STUDENTS' USE OF VISUALIZATIONS WHEN SOLVING VOLUME PROBLEMS 171
Krista Kay Bresock, Vicki Lynn Sealey

A COMPARISON OF APPROACHES TO STIMULATED RECALL INTERVIEWS WITH MATHEMATICS TEACHERS IN ORDER TO IDENTIFY SHIFTS IN ATTENTION 179
Julian Tom Brown

DECISION-MAKING IN NOTICING STUDENTS' PROPORTIONAL REASONING 187
Àngela Buforn, Ceneida Fernández, Salvador Llinares

CLUMPS OR CHUNKS? - CONTEXTUAL RELEVANCE OF STUDENTS' FEATURES OF THE DATA 195
Christian Büscher

A LARGE SCALE CASCADE MODEL IN THE CONTEXT OF MATHEMATICS CURRICULUM REFORM: INTERACTIVE FACTORS OF INFLUENCE ON MULTIPLIERS' WORK 203
Ana Paula Canavarro, Leonor Santos

RESPONSE PROCESS VALIDITY EVIDENCE: A PROPORTIONAL REASONING EXAMPLE 211
Michele Carney, Katie Paulding

MATHEMATICAL INDUCTION AT THE TERTIARY LEVEL: LOOKING BEHIND APPEARENCES 219
Gemma Carotenuto, Cristina Coppola, Pietro Di Martino

AN INNOVATIVE QUALITATIVE VIDEO ANALYSIS INSTRUMENT TO ASSESS THE QUALITY OF POST-SECONDARY ALGEBRA INSTRUCTION 227
Anne Cawley, Irene Duranczyk, Angeliki Mali, Vilma Mesa, April Strom, Laura Watkins, Patrick Kimai, Dexter Lim

THE PREDICTIVE ROLE OF DIFFERENT REASONING FORMS TO STUDENTS' EARLY ALGEBRAIC THINKING ABILITIES 235
Maria Chimoni, Demetra Pitta-Pantazi

PEER OBSERVATION AS A TOOL TO FACILITATE MATHEMATICS TEACHERS' SELF-REFLECTION IN A PROFESSIONAL DEVELOPMENT INTERVENTION 243
Brantina Chirinda, Patrick Barmby

A SOCIAL EXPRESSION OF NUMBER USING TOUCHCOUNTS 251
Sean Chorney

THE DESIRED TEACHER REFLECTED IN RESEARCH ARTICLES ON PRACTICUM 259
Iben Maj Christiansen, Lisa Österling

THE NATURAL NUMBER BIAS IN ARITHMETIC OPERATIONS: THE CASE OF THE REPRESENTATIONAL FORM OF THE NUMBERS 267
Konstantinos Christou

RE-THINKING 'CONCRETE TO ABSTRACT': TOWARDS THE USE OF SYMBOLICALLY STRUCTURED ENVIRONMENTS 275
Alf Coles, Nathalie Sinclair

TEACHER BELIEFS AND SUPPORT FOR ARGUMENTATION 283
AnnaMarie Conner, Laura Marie Singletary

FINANCIAL LITERACY: PRACTICING MONEY MANAGEMENT AS YOUR FUTURE-SELF 291
Mary Connolly, Cynthia Nicol

BOUNDARY CROSSING IN DESIGN BASED RESEARCH - LESSONS LEARNED FROM TAGGING DIDACTIC METADATA 299
Jason Cooper, Shai Olsher

JOURDAIN AND DIENES EFFECTS REVISITED - PLAYING TIC TAC TOE OR LEARNING NON-EUCLIDEAN GEOMETRY?...................... 307
Jason Cooper, Alon Pinto

FROM ‘FROWNS AND GROANS’ TO ‘ASTONISHMENT AND DELIGHT’: SEEKING INDICATORS OF A MATHEMATICS TEACHERS IDENTITY 315
Cosette Crisan, Melissa Rodd

LINKING THEORY AND PRACTICE: PROSPECTIVE TEACHERS CREATING FICTIONAL CLASSROOM DISCUSSIONS 323
Annalisa Cusi, Francesca Morselli

ELEMENTARY STUDENTS' CONDITIONAL REASONING SKILLS IN MATHEMATICAL AND EVERYDAY CONTEXTS...................... 331
Anastasia Datsogianni, Stefan Ufer, Beate Sodian

THE FIRST-TIME PHENOMENON: SUCCESSFUL STUDENTS' MATHEMATICAL CRISIS IN SECONDARY-TERTIARY TRANSITION 339
Pietro Di Martino, Francesca Gregorio

ROLE OF USING AN ALTERNATIVE CONCEPT DEFINITION IN CONDUCTING MATHEMATICAL TASKS OF TEACHING: THE CASE OF EXPLAINING WHY AN ALGORITHM WORKS 347
Merve Dilberoglu, Cigdem Haser

CONCEPTUALIZING AN EXPERT TEACHER’S EXPERTISE IN A LESSON DESIGN STUDY IN SHANGHAI 355
Liping Ding, Keith Jones

MATHEMATICIANS' CRITERIA FOR ACCEPTING THEOREMS AND PROOFS – AN INTERNATIONAL STUDY 363
Anika Dreher, Aiso Heinze

FROM EVERYDAY PROBLEM TO A MATHEMATICAL SOLUTION — UNDERSTANDING STUDENT REASONING BY IDENTIFYING THEIR CHAIN OF REFERENCE 371
Jonas Dreyøe, Dorte Moeskær Larsen, Morten Misfeldt

CONCEPTUALISING TRANSLATIONS BETWEEN
REPRESENTATIONS 379
Anneli Dyrvold

STUDENTS' CONCEPTIONS FOR CURVE LENGTH: A
HYPOTHETICAL LEARNING TRAJECTORY APPROACH 387
Cheryl Lynn Eames

PREDICTORS OF DEMAND FOR MATHEMATICS SUPPORT 395
Antony Edwards, David Carroll

ONE STUDENT'S DISCURSIVE DEVELOPMENT ON
ROTATION IN RELATION TO INSTRUCTION FROM
A COMMOGNITIVE PERSPECTIVE 403
Elcin Emre-Akdogan, Beste Gucler, Ziya Argun

FROM PRINCIPLES OF VISION AND DIVISION TO A SYSTEM
PREMISED ON AND SUBJECT TO INTERANIMATED DIMENSIONS:
SOME REFLECTIONS ON IDENTITY IN MATHEMATICS
EDUCATION 411
Osnat Fellus

THINKING IN MOVEMENT AND MATHEMATICS: A CASE STUDY 419
Francesca Ferrara, Giulia Ferrari

USE OF STUDENT-PRODUCED VIDEOS IN THE
TEACHING OF COMBINATORICS 427
Anders Støle Fidje

WHAT VOCABULARY DO TEACHERS USE WHEN ANALYSING
THE USE OF REPRESENTATIONS IN CLASSROOM SITUATIONS? 435
Marita Friesen, Carmel Mesiti, Sebastian Kuntze

SOLVING ARITHMETIC-ALGEBRAIC WORD PROBLEMS
BY 10- TO 12-YEAR-OLD STUDENTS 443
Torsten Fritzlar, Nadja Karpinski-Siebold

ACADEMIC PROCRASTINATION IN THE TRANSITION FROM
SCHOOL TO UNIVERSITY MATHEMATICS 451
Sebastian Geisler, Katrin Rolka

GAP AND CONGRUENCY EFFECT IN FRACTION COMPARISON 459
Juan Manuel González-Forte, Ceneida Fernández, Wim Van Dooren

GENDER SPECIFICITIES IN A SUPPORT PROJECT
FOR ENGINEERING STUDENTS 467
Birgit Griese

TEACHER'S GAZE BEHAVIOR WHEN SCAFFOLDING PEER INTERACTION AND MATHEMATICAL THINKING DURING COLLABORATIVE PROBLEM-SOLVING ACTIVITY 475
Eeva Haataja, Enrique Garcia Moreno-Esteva,
Miika Toivanen, Markku S. Hannula

INDEX

INDEX OF AUTHORS AND CO-AUTHORS (VOLUME 2) 485

TABLE OF CONTENTS

VOLUME 3

RESEARCH REPORTS HAM-PI

FRAMING THE SOCIAL DYNAMICS OF SMALL GROUP WORK IN ADOLESCENCE AS PEER CULTURES OF EFFORT AND ACHIEVEMENT 3
Jill Hamm, Abigail Hoffman, Kerrilyn Lambert, Daniel Heck

WHICH IS SMALLER...? PARTIAL UNDERSTANDINGS AND MISCONCEPTIONS ABOUT MULTIPLICATION AND DIVISION BY FRACTIONS 11
Pircha Hamo, Bat-Sheva Ilany, Meir Buzaglo

DOCTORAL PROGRAMS' CONTRIBUTION TO BECOMING A MATHEMATICS EDUCATION RESEARCHER 19
Çiğdem Haser

INSTRUMENT TO ANALYSE DYADS' COMMUNICATION AT TERTIARY LEVEL 27
Mathias Hattermann, Daniel Heinrich, Alexander Salle, Stefanie Schumacher

STRIVING FOR EQUITY: HOW POLICY SHAPES OUR UNDERSTANDING OF RACE IN MATH CLASS 35
Michelle Christine Hawks

CAN STUDENTS CONSTRUCT NON-CONSTRUCTIVE REASONING? IDENTIFYING FUNDAMENTAL SITUATIONS FOR PROOF BY CONTRADICTION 43
Toru Hayata, Yusuke Uegatani, Ryoto Hakamata

MATHEMATICS DISCOURSE IN SMALL GROUPS 51
Daniel J. Heck, Pippa Hoover, Jessica Porter, Jill V. Hamm

THE RELATION OF CHILDREN'S PERFORMANCES IN SPATIAL TASKS AT TWO DIFFERENT SCALES OF SPACE 59
Cathleen Heil

WHICH ESTIMATION SITUATIONS ARE RELEVANT FOR A VALID ASSESSMENT OF MEASUREMENT ESTIMATION SKILLS? 67
Aiso Heinze, Dana Farina Weiher, Hsin-Mei Huang, Silke Ruwisch

RESPONDING TO TEACHERS: LEARNING HOW TO USE VERBAL METACOMMUNICATION AS A MATHEMATICS TEACHER EDUCATOR 75
Tracy Helliwell

THE ROLE OF FINGER GNOSIS IN THE DEVELOPMENT OF EARLY NUMBER SKILLS ... 83
Caroline Ann Hilton

PROFESSIONALISATION OF PROSPECTIVE TEACHERS THROUGH THE PROMOTION OF COGNITIVE DIAGNOSTIC COMPETENCE ... 91
Natalie Hock, Rita Borromeo Ferri

WHAT DO MATHEMATICIANS WISH TO TEACH TEACHERS IN SECONDARY SCHOOL ABOUT MATHEMATICS? ... 99
Anna Hoffmann, Ruhama Even

GESTURES AS EMBODIMENTS OF VARIABLES AND ALGEBRAIC EXPRESSIONS ... 107
Mirjana Hotomski

SOLVING COMBINATORIAL COUNTING PROBLEMS: PRIMARY CHILDREN’S RECURSIVE STRATEGIES ... 115
Karina Höveler

PRE-SERVICE MATHEMATICS TEACHERS’ WHOLE-CLASS DIALOGS DURING FIELD PRACTICE ... 123
Siri-Malén Høynes, Torunn Klemp, Vivi Nilssen

INVESTIGATING SECONDARY SCHOOL STUDENTS' EPISTEMOLOGIES THROUGH A CLASS ACTIVITY CONCERNING INFINITY ... 131
Paola Iannone, Davide Rizza, Athina Thoma

WATCHING MATHEMATICIANS READ MATHEMATICS ... 139
Matthew Inglis, Lara Alcock

ENHANCING TEACHER NOTICING USING A HYPOTHETICAL LEARNING TRAJECTORY ... 147
Pedro Ivars, Ceneida Fernández, Salvador Llinares, Ban Heng Choy

USING EQUATIONS TO DEVELOP A COHERENT APPROACH TO MULTIPLICATION AND MEASUREMENT ... 155
Andrew Gyula Izsak, Sybilla Beckmann

A WRITTEN, LARGE-SCALE ASSESSMENT MEASURING GRADATIONS IN STUDENTS’ MULTIPLICATIVE REASONING ... 163
Heather Lynn Johnson, Ron Tzur, Nicola M Hodkowski, Cody Jorgensen, Bingqian Wei, Xin Wang, Alan Davis

THEORIES ABOUT MATHEMATICAL CREATIVITY IN CONTEMPORARY RESEARCH: A LITERATURE REVIEW ... 171
Julia Joklitschke, Benjamin Rott, Maike Schindler

20 YEARS OF MATHEMATICS MOTIVATION MIRRORED THROUGH TIMSS: EXAMPLE OF NORWAY ... 179
Hege Kaarstein, Jelena Radišić, Trude Nilsen

CRITERIA FOR KNOWING A GEOMETRICAL OBJECT:
THE ENACTIVIST PERSPECTIVE 187
Kazuya Kageyama

MICRO-EVOLUTION OF DOCUMENTATIONAL WORK IN THE
TEACHING OF THE VOLUME OF REVOLUTION 195
Lina Kayali, Irene Biza

SUPPORTING PRESERVICE TEACHERS'
IN-THE-MOMENT NOTICING 203
Hulya Kilic, Oguzhan Dogan, Sena Simay Tun, Nil Arabaci

CORE MATHEMATICAL TEACHING PRACTICES IN ALGEBRAIC
AND FUNCTIONAL RELATIONS 211
Hee-jeong Kim, Ji-Won Son

TEACHER CAPACITY FOR PRODUCTIVE RESOURCES USE 219
Ok-Kyeong Kim

SECONDARY SCHOOL STUDENTS' APPRAISAL OF
MATHEMATICAL PROOFS 227
Kotaro Komatsu, Miho Yamazaki, Taro Fujita, Keith Jones, Naoki Sue

LEARNING MATHEMATICS THROUGH ONLINE FORUMS:
A CASE OF LINEAR ALGEBRA 235
Igor' Kontorovich

ACTIVATION AND MONITORING OF PRIOR MATHEMATICAL
KNOWLEDGE IN MODELLING PROCESSES 243
Janina Krawitz, Stanislaw Schukajlow

PRIMARY STUDENT'S DATA-BASED ARGUMENTATION – AN
EMPIRICAL REANALYSIS 251
Jens Oliver Krummenauer, Sebastian Kuntze

FINNISH PRIMARY TEACHERS' INTERACTION WITH
CURRICULUM MATERIALS – DIGITALISATION AS AN
AUGMENTING ELEMENT 259
Heidi Krzywacki, Kirsti Hemmi, Janine Remillard, Hendrik van Steenbrugge

USING FINGERS TO DISCERN THE STRUCTURE OF
PART-WHOLE RELATIONS OF NUMBERS IN PRESCHOOL 267
Angelika Kullberg, Camilla Björklund, Irma Brkovic, Ulla Runesson Kempe

TEACHERS' CRITERION AWARENESS AND THEIR
ANALYSIS OF CLASSROOM SITUATIONS 275
Sebastian Kuntze, Marita Friesen

PRIMARY GRADE STUDENTS' FUNDAMENTAL IDEAS OF
GEOMETRY REVEALED VIA DRAWINGS 283
Ana Kuzle, Dubravka Glasnović Gracin, Martina Klunter

CONNECTED WORKING SPACES: DESIGNING AND EVALUATING MODELLING-BASED TEACHING SITUATIONS 291
Jean-baptiste Lagrange

THE INFLUENCE OF SALIENCY IN INTUITIVE REASONING 299
Stephanie Lem, Wim Van Dooren

IS MATHEMATICAL CREATIVITY RELATED TO MATHEMATICAL EXCELLENCE? TEACHERS' BELIEFS 307
Esther Levenson

EARLY MATHEMATICAL REASONING – THEORETICAL FOUNDATIONS AND POSSIBLE ASSESSMENT 315
Anke Lindmeier, Esther Brunner, Maike Grüßing

HOW DRAGGING MEDIATES A DISCOURSE ABOUT FUNCTIONS 323
Giulia Lisarelli

MATHEMATICS TEACHERS' IDENTITY DEVELOPMENT IN THE CONTEXT OF PROFESSIONAL MASTER'S DEGREES 331
Leticia Losano, Dario Fiorentini

CHANGES IN ATTITUDES REVEALED THROUGH STUDENTS' WRITING ABOUT MATHEMATICS 339
Wes Maciejewski

WHICH KEY MEMORABLE EVENTS ARE EXPERIENCED BY STUDENTS DURING CALCULUS TUTORIALS? 347
Ofer Marmur, Boris Koichu

ANALYSIS OF THE MATHEMATICAL DISCOURSE OF UNIVERSITY STUDENTS WHEN DESCRIBING AND DEFINING GEOMETRICAL FIGURES 355
Verónica Martín-Molina, Rocío Toscano, Alfonso J. González-Regaña, Aurora Fernández-León, José María Gavilán-Izquierdo

LINKING INFORMAL AND FORMAL MATHEMATICAL REASONING: TWO DIRECTIONS ACROSS THE SAME BRIDGE? 363
Jake McMullen, Lauren B. Resnick

STUDENTS' SENSE OF BELONGING TO MATHEMATICS IN THE SECONDARY-TERTIARY TRANSITION 371
Maria Meehan, Emma Howard, Aoibhinn Ni Shuilleabhain

THE PROFESSIONAL, PEDAGOGICAL LANGUAGE OF MATHEMATICS TEACHERS: A CULTURAL ARTEFACT OF SIGNIFICANT VALUE TO THE MATHEMATICS COMMUNITY 379
Carmel Mesiti, David Clarke

YOU SEE (MOSTLY) WHAT YOU PREDICT: THE POWER OF GEOMETRIC PREDICTION 387
Elisa Miragliotta, Anna Baccaglini-Frank

PROOF AND PROVING IN HIGH SCHOOL GEOMETRY: A TEACHING EXPERIMENT BASED ON TOULMIN'S SCHEME 395
Andreas Moutsios-Rentzos, Ioanna Micha

STUDENTS' PATHWAYS FOR SOLVING PROBABILITY PROBLEMS 403
Lydia Mutara, Judah Paul Makonye

THE MATHEMATICS TEXTBOOK FOR RURAL POPULATION IN BRAZIL: LEARNING TO BE A MODERNIZED FARMER 411
Vanessa Franco Neto, Paola Valero

COGNITIVE ABILITIES AND MATHEMATICAL REASONING IN PRACTICE AND TEST SITUATIONS 419
Mathias Norqvist

ARE ADULTS BIASED IN COMPLEX FRACTION COMPARISON, AND CAN BENCHMARKS HELP? 427
Andreas Obersteiner, Martha W. Alibali

SECOND-GRADERS' PREDICTIVE REASONING STRATEGIES 435
Gabrielle Ruth Oslington

PRODUCTIVE WAYS OF ORGANISING PRACTICUM – WHAT DO WE KNOW? A SYSTEMATIC REVIEW 443
Lisa Österling, Iben Maj Christiansen

THE USE OF 'MENTAL' BRACKETS WHEN CALCULATING ARITHMETIC EXPRESSIONS 451
Ioannis Papadopoulos, Robert Gunnarsson

MAKING MATHEMATICAL LEARNING LONG-TERMED AND EFFECTIVE USING INTERLEAVED PRACTICES 459
Stella Pede, Rita Borromeo Ferri, Frank Lipowsky

PROSPECTIVE PRIMARY TEACHERS' CONCEPTUAL UNDERSTANDING OF MATHEMATICAL PROBLEMS AND PROBLEM SOLVING 465
Juan Luis Piñeiro, Elena Castro-Rodríguez, Enrique Castro

"I ALWAYS WISHED THAT I HAD A MATHEMATICAL MIND": MATHEMATICAL ABILITY AND OTHER STORIES 473
Dionysia Pitsili-Chatzi

INDEX

INDEX OF AUTHORS AND CO-AUTHORS (VOLUME 3) 483

TABLE OF CONTENTS

VOLUME 4

RESEARCH REPORTS PR-Z

EXPLORING PERSPECTIVES ON MATHEMATICAL MODELLING: A LITERATURE SURVEY 3
Armando Paulino Preciado Babb, Armando Solares Rojas, Fredy Peña, Andrea Ortiz, Marisol Sandoval Rosas, Remedios Soriano Velasco, Vicente Carrión Vázquez, Mauricio Farrugía Fuentes

ARE TEACHERS' LANGUAGE VIEWS CONNECTED TO THEIR DIAGNOSTIC JUDGMENTS ON STUDENTS' EXPLANATIONS? 11
Susanne Prediger, Dilan Şahin-Gür, Carina Zindel

PROBLEM SOLVING: HOW PRESERVICE TEACHERS UNDERSTAND IT DURING THEIR PRESERVICE LEARNING 19
Jamie S Pyper, Stephen MacGregor

A STUDY OF MIDDLE SCHOOL STUDENTS' ALGEBRAIC PROOFS IN CHINA 27
Xuefeng Qiao

SECONDARY PROSPECTIVE TEACHERS' INTERPRETATIVE KNOWLEDGE ON A MEASUREMENT SITUATION 35
Miguel Ribeiro, Arne Jakobsen, Maria Mellone

GEOMETRY IN KINDERGARTEN: FIRST STEPS TOWARDS THE DEFINITION OF CIRCUMFERENCE 43
Elisabetta Robotti

USING CRITICAL EVENTS IN PRE-SERVICE TRAINING: EXAMINING THE COHERENCE LEVEL BETWEEN INTERPRETATIONS OF STUDENTS' MATHEMATICAL THINKING AND INTERPRETATIONS OF TEACHER'S RESPONSES 51
Sigal Rotem, Michal Ayalon

CREATIVITY OR IMAGINATION: CHALLENGES WITH MEASURING CREATIVITY 59
Benjamin Rott, Peter Liljedahl

TENSIONS IN IMPLEMENTING MATHEMATICS JOURNALING 67
Annette Rouleau

FROM "HOW GOOD I AM!" TO "FORGIVE ME…PLEASE TRUST ME"- MICROAGGRESSIONS AND ANGLES 75
Ulrika Ryan

SOME CHANGES OF MATH ANXIETY GROUPS BASED ON TWO MEASUREMENTS, MASS & EEG 83
Byeongguk Ryoo, Sang Sook Choi-Koh

TRANSFORMATION OF A GEOMETRIC DIAGRAM TO PRODUCE A CONJECTURE AND ITS PROOF 91
Adalira Saenz-Ludlow, Anna Athanasopoulou

"GROWTH GOES DOWN, BUT OF WHAT?" A CASE STUDY ON LANGUAGE DEMANDS IN QUALITATIVE CALCULUS 99
Dilan Sahin-Gür, Susanne Prediger

USE OF A LEARNING TRAJECTORY AS A CONCEPTUAL INSTRUMENT TO DEVELOP THE COMPETENCE OF PROFESSIONAL NOTICING 107
Gloria Sanchez-Matamoros, Mar Moreno, Julia Valls, M Luz Callejo

EYE-TRACKING FOR STUDYING MATHEMATICAL DIFFICULTIES—ALSO IN INCLUSIVE SETTINGS........ 115
Maike Schindler, Achim J. Lilienthal

A TEACHER'S REFLECTIVE PROCESS IN A VIDEO-BASED PROFESSIONAL DEVELOPMENT PROGRAM 123
Gil Schwarts, Ronnie Karsenty

MIDDLE SCHOOL STUDENTS' REASONING ABOUT VOLUME AND SURFACE AREA 131
Rebecca Seah, Marj Horne

USING SELF-VIDEO-BASED CONVERSATIONS IN TRAINING MATHEMATICS TEACHER INSTRUCTORS........ 139
Ruti Segal, Yaron Lehavi, Avi Merzel, Ami Baram, Bat-Sheva Eylon

ANALYSIS OF THE PCK OF AN ELEMENTARY MATHEMATICS MASTER TEACHER 147
Eun-Min Seo, Jin-Ho Kim, Woong Lim

EXAMINING TEACHERS' DISCOURSE ON STUDENTS' STRUGGLE THROUGH FIGURED WORLDS 155
Galit Shabtay, Einat Heyd-Metzuyanim

GROUP THINKING STYLES AND THEIR MODELLING PROCESS WHILE ENGAGING IN MODELLING ACTIVITIES........ 163
Juhaina Awawdeh Shahbari, Rania Salameh

A DUAL EYE-TRACKING STUDY OF OBJECTIFICATION AS STUDENT–TUTOR JOINT ACTIVITY APPROPRIATION........ 171
Anna Shvarts

TEXTBOOK EFFECTS ON THE DEVELOPMENT OF ADAPTIVE EXPERTISE 179
Henning Sievert, Ann-Kathrin Van Den Ham, Inga Niedermeyer, Aiso Heinze

BRAZILIAN HIGH SCHOOL TEXTBOOKS: MATHEMATICS AND STUDENTS' SUBJECTIVITY 187
Marcio Silva, Paola Valero

DESIGNING FOR GUIDED REINVENTION OF MATHEMATICAL CONCEPTS 195
Martin Simon

STRUCTURING STUDENTS' MATHEMATICAL CONVERSATIONS WITH FLOWCHARTS AND INTENTION ANALYSIS – AFFORDANCES AND CONSTRAINTS 203
Joakim Vilhelm Smedlund, Kirsti Hemmi, Ann-Sofi Röj-Lindberg

SCHOOLS' STRATEGIES FOR PROMOTING GIRLS' PARTICIPATION IN MATHEMATICS 211
Cathy Anne Smith, Jennie Golding

EFFECTS OF SOCIOECONOMIC STATUS ON MIDDLE SCHOOL STUDENTS' MATHEMATICS ACHIEVEMENT IN CHINA 219
Shuang Song, Kan Guo, Yiming Cao

WICKED PROBLEMS IN SCHOOL MATHEMATICS 227
Lisa Steffensen, Rune Herheim, Toril Eskeland Rangnes

OBJECTIFING TINGA: A CASE OF CHILDREN INVENTING THEIR OWN DISCOURSE ON FRACTIONAL QUANTITIES 235
Aya Steiner

INVESTIGATING SELF-EFFICACY EXPECTATIONS AND MASTERY EXPERIENCES ACROSS A SEQUENCE OF LESSONS IN MATHEMATICS 243
Karin Elisabeth Sørlie Street, Lars-Erik Malmberg, Gabriel J. Stylianides

MATHEMATICS IN DISGUISE: EFFECTS OF THE EXTERNAL CONTEXT OF MATHEMATICAL WORD PROBLEMS 251
Anselm R. Strohmaier, Kristina M. Reiss

TASK DESIGN TO PROMPT MAKING SENSE OF PRE-CALCULUS CONCEPTS USING DYNAMIC TECHNOLOGICAL TOOLS 259
Osama Swidan

MAKING SENSE OF THE TEACHING OF CALCULUS FROM A COMMOGNITIVE PERSPECTIVE 267
Innocent Tasara

KINDERGARTEN TEACHERS' KNOWLEDGE OF STUDENTS: THE CASE OF REPEATING PATTERNS 275
Dina Tirosh, Pessia Tsamir, Esther Levenson, Ruthi Barkai

THE USE OF NONSTANDARD PROBLEMS IN AN ODE COURSE FOR ENGINEERING STUDENTS 283
Stephanie Treffert-Thomas, Svitlana Rogovchenko, Yuriy Rogovchenko

ENACTMENT OF INQUIRY-BASED MATHEMATICS TEACHING AND LEARNING: THE CASE OF STATISTICAL ESTIMATION 291
Chrissavgi Triantafillou, Giorgos Psycharis, Dionysia Bakogianni, Despina Potari

THE INTERPLAY OF INFORMATIVE ASSESSMENT CRITERIA AND CONTINUOUS FEEDBACK WITH MATHEMATICS STUDENTS' LEARNING ORIENTATIONS 299
Laura Tuohilampi, Juuso Henrik Nieminen, Jokke Häsä, Johanna Rämö

VARIATION OF STUDENT ENGAGEMENT BETWEEN DIFFERENT ALGEBRA TASKS 307
Tatu Dimitri Tuomela, Markus Hähkiöniemi

INTERPLAY IN STUDENTS' THINKING MODES AND REPRESENTATION TYPES OF LINEAR ALGEBRA IN A DGS 315
Melih Turgut

SAME UNIT COORDINATION: A CONCEPTUAL SCREENER FOR MIXED UNIT COORDINATION AND BASE-10, PLACE VALUE REASONING 323
Ron Tzur, Bingqian Wei, Amy Smith, Anderson Norton, Alan Davis, Heather Lynn Johnson

FACILITATING CONCEPTUAL ENGAGEMENT WITH FRACTIONS THROUGH SUSPENDING THE USE OF MATHEMATICAL TERMINOLOGY 331
Pamela Vale, Mellony Graven

INVESTIGATING LEARNERS' FRACTION UNDERSTANDING: A LONGITUDINAL STUDY IN UPPER ELEMENTARY SCHOOL 339
Jo Van Hoof, Tine Degrande, Eva Ceulemans, Lieven Verschaffel

EMERGENT PROPORTIONAL REASONING: SEARCHING FOR EARLY TRACES IN FOUR- TO FIVE-YEAR OLDS 347
Elien Vanluydt, Lieven Verschaffel, Wim Van Dooren

COUNTING ON – LONG TERM EFFECTS OF AN EARLY INTERVENTION PROGRAMME 355
Helena Vennberg, Mathias Norqvist

RITUALISED AND EXPLORATORY GRAPHING ROUTINES IN MATHEMATICAL MODELLING: THE DIGOXIN TASK 363
Olov Viirman, Elena Nardi

SYMBOLISATION AND OBJECTIFICATION THROUGH SOCIAL INTERACTIONS FOR MEANINGFUL LEARNING OF MATHEMATICS 371
Joelle Vlassis, Isabelle Demonty

THE APPLIED KNOWLEDGE OF TRAINEES AS INDUSTRIAL CLERKS SOLVING PROBLEMS WITH VOCATIONAL AND NON-VOCATIONAL CONTEXT 379
Robert von Hering, Anja Rietenberg, Aiso Heinze, Anke Lindmeier

STUDENTS' READINESS TO APPROPRIATE THE DERIVATIVE - META-KNOWLEDGE AS SUPPORT FOR THE ZPD 387
Pauline Vos, Gerrit Roorda

CROSSING THE BOUNDARIES BETWEEN SCHOOL MATHEMATICS AND WORKPLACE THROUGH AUTHENTIC TASKS 395
Nikolaos Vroutsis, Giorgos Psycharis, Chrissavgi Triantafillou

DOES THE COGNITIVE DEMAND OF A PROBLEM INCREASE WHEN THE ANSWER IS AN INEQUALITY? 403
Ilana Waisman, Mark Leikin, Roza Leikin

A CONCEPT ANALYSIS OF THE NOTION CONCEPT: CONTRIBUTIONS OF AN ANALYSING TOOL 411
Lotta Wedman

THE DEVELOPMENT OF THE CONCEPT OF LIMIT – ASPECTS AND BASIC MENTAL MODELS 419
Hans-Georg Weigand

EXAMINING EXPLORATIVE INSTRUCTION ACCORDING TO THE REALIZATION TREE ASSESSMENT TOOL 427
Merav Weingarden, Einat Heyd-Metzuyanim

ACTION STRATEGIES IN SPATIAL GEOMETRY PROBLEM SOLVING SUPPORTED BY DYNAMIC GEOMETRY SOFTWARE 435
Mirela Widder, Abraham Berman, Boris Koichu

MATHEMATICS ACHIEVEMENT AND THE ROLE OF WORKING MEMORY AND ATTENTION - EVIDENCE FROM A LARGE-SCALE STUDY WITH FIRST GRADERS 443
Kirsten Winkel, Henning Mueller, Daniel Schunk

ELEMENTARY AND SECONDARY STUDENTS' FUNCTIONAL THINKING WITH TABLES AND DIAGRAMS 451
Ulises Xolocotzin, Matthew Inglis, Ana Medrano

USING THREE FIELDS OF EDUCATION RESEARCH TO FRAME THE DEVELOPMENT OF DIGITAL GAMES 459
Michelle Zandieh, Caro Williams-Pierce, David Plaxco, Ashish Amresh

UNFOLDING AND COMPACTING WHEN CONNECTING REPRESENTATIONS OF FUNCTIONS 467
Carina Zindel

THE ROLE OF HORIZON CONTENT KNOWLEDGE IN TEACHERS' RECOGNITION AND INTERPRETATION OF STUDENTS' MATHEMATICAL MISCONCEPTIONS 475
Sotirios Zoitsakos, Theodosios Zachariades, Charalampos Sakonidis

INDEX

INDEX OF AUTHORS AND CO-AUTHORS (VOLUME 4) 485

TABLE OF CONTENTS

VOLUME 5

ORAL COMMUNICATIONS

EVOLUTION OF UNDERGRADUATE STUDENTS' MATHEMATICAL REASONING 3
Aaltje Berendina Aaten, Gerrit Roorda, Johan Deprez, Martin Goedhart

PRE-CALCULUS COLLEGE STUDENTS' QUANTITATIVE REASONING IN THE CONTEXT OF REAL-WORLD SITUATIONS 4
Ayça Akın, H. Seda Sezgin

WHY DO INSTRUCTORS PREFER NOT TO PRESENT PROOFS OF SOME THEOREMS? 5
Esra Aksoy, Serkan Narlı

ELEMENTARY PRE-SERVICE MATHEMATICS TEACHERS' CONCEPTIONS OF ALGEBRA 6
Burcu Alapala, Isil Isler-Baykal

GRICE'S RELEVANCE MAXIM FOR INVESTIGATING UNDERGRADUATES' ARGUMENTATIONS 7
Giovannina Albano, Pier Luigi Ferrari, Anna Pierri

CHALLENGES OF MATHEMATICS TEACHERS WITH PROOF: OSCILLATING BETWEEN DIFFERENT EXPECTATIONS 8
Sikunder Ali, Trond Stølen Gustavsen, Sigurd Hals, Andrea Hofmann

CROSS-SECTIONAL ANALYSIS OF STUDENTS' ANSWERS TO A REALISTIC WORD PROBLEM FROM GRADE 2 TO 10 9
Gabriella Ambrus, Eszter Herendiné Kónya, Zoltán Kovács, Judit Szitányi, Eszter Varga, Csaba Csíkos

GENDER ISSUES IN THE KANGAROO CONTEST 10
Mark Applebaum, Viktor Freiman

MATHEMATICS TEACHER'S PROBLEMS IN THE PROCESS OF PEDAGOGICAL REASONING FROM THE PERSPECTIVE OF CURRICULUM MAKER; THE CASE OF AN ELEMENTARY SCHOOL TEACHER IN THE PHILIPPINES 11
Mitsue Arai

STRATEGIES OF SOLUTION OF GEOMETRIC PATTERN PROBLEMS AS TRAITS OF MATHEMATICAL GIFTEDNESS 12
Eva Arbona, Angel Gutierrez, María José Beltrán-Meneu

APPLYING JAPANESE DIDACTIC PRACTICES IN A SWEDISH CLASSROOM 13
Yukiko Asami-Johansson

SIMILARITIES BETWEEN MATHEMATICAL PROBLEMS FROM THE PERSPECTIVE OF PRIMARY STUDENTS 14
Daniela Assmus, Torsten Fritzlar, Frank Förster

TEXTBOOK APPROACHES AND PISA EXAM QUESTIONS 15
Weverton Ataide Pinheiro, Kin Hang Lei, Feng-Lin Lu, Tai-Yih Tso

REPRESENTATION OF INDETERMINATE QUANTITIES IN FUNCTIONAL CONTEXTS BY THIRD GRADE STUDENTS 16
Cristina Ayala-Altamirano, Marta Molina

THE COMPLEXITIES OF ENACTING MULTIPLE REPRESENTATIONS WHEN TEACHING FRACTIONS 17
Fay Baldry

ANALYSIS OF MATHEMATICALLY GIFTED STUDENTS' ANSWERS TO COGNITIVELY DEMANDING SCHOOL TASKS 18
Clara Benedicto Baldonado, Ángel Gutiérrez, Adela Jaime

READING STYLES: GOALS, PRIOR KNOWLEDGE AND TEXT 19
Margot Berger

HOW TO OVERCOME FRAGMENTATION IN PRE-SERVICE TEACHER EDUCATION IN MATHEMATICS AT UNIVERSITY LEVEL 20
Angelika Bikner-Ahsbahs, Erik Hanke, Ingolf Schäfer

PROFESSIONAL ONLINE LEARNING COMMUNITIES IN MATHEMATICS: A CASE STUDY OF THE ISRAELI VHS 21
Yaniv Biton, Osnat Fellus

PEER TUTORS' AWARENESS OF AFFECTIVE FACTORS 22
Katie Bjorkman

PROSPECTIVE TEACHERS' EXPLANATIONS FOR INTEGER WORD PROBLEMS 23
Laura Bofferding, Nicole Wessman-Enzinger

CHILDREN AND ADULTS UNDERSTANDING RANDOMNESS, SAMPLE SPACE AND COMPARISON OF PROBABILITIES 24
Rute Elizabete Borba, Rita Batista, Ewellen Lima

ON STUDENTS' UNDERSTANDING OF IMPLICIT DIFFERENTIATION, BASED ON TWO LENSES: APOS AND OSA 25
Vahid Borji, Vicenç Font, Hassan Alamolhodaei,
Alicia Sánchez, Luis R. Pino-Fan

AN EXAMINATION OF A PRESERVICE MATHEMATICS TEACHER'S REASONING ON CONSTRUCTION TASKS IN A DYNAMIC GEOMETRY ENVIRONMENT 26
Gulay Bozkurt, Candas Uygan, Melih Turgut

SECONDARY STUDENTS' LOGICAL REASONING ABILITIES 27
Hugo Bronkhorst, Gerrit Roorda, Cor Suhre, Martin Goedhart

RESEARCH IN MATHEMATICS EDUCATION SEEN AS AN ACTIVITY: TEACHER PROFESSIONAL DEVELOPMENT IN FOCUS 28
Wellington LIma Cedro

A SECONDARY SCHOOL MATHEMATICS TEACHER'S PROFESSIONAL DEVELOPMENT IN DIFFERENTIATED INSTRUCTION: A CASE STUDY 29
Chang-Hua Chen

HOW TO HELP AN EXPERIENCED TEACHER TO TAKE THE FIRST STEP SUCCESSFULLY IN CHANGING THEIR MATHEMATICS TEACHING 30
Ying-Hao Cheng, Hui-Yu Hsu, Jian-Cheng Chen

IMPORTANT FACTORS CLASSIFYING MACAO'S RESILIENT AND NON-RESILIENT STUDENTS IN PISA 2012 MATHEMATICAL STUDY FOR INFORMED POLICY MAKING 31
Kwok-cheung Cheung, Man-kai Ieong, Soi-kei Mak, Pou-seong Sit

IMPLEMENTING STANDARDS-BASED MATHEMATICS: TOWARD IMPROVING CONCEPTUAL UNDERSTANDING FOR MIDDLE SCHOOL IMMIGRANT STUDENTS 32
Sousada Chidthachack

EXAMINING STUDENTS' INFORMAL MODELING STRATEGIES TO SOLVE APPLIED MATHEMATICS PROBLEMS 33
Victor Cifarelli

PROMOTING MATHEMATICS' TEACHING THROUGH HANDS-ON SCIENCE EXPERIMENTS 34
Maria Cristina Costa, António Domingos

AN EXPLORATION OF HIGH SCHOOL FRENCH IMMERSION STUDENTS' COMMUNICATION DURING A COLLABORATIVE MATHEMATICS PROBLEM-SOLVING TASK 35
Karla Marie Culligan

A TEST TO MEASURE EARLY NUMBER SKILLS PROGRESS AMONG 4 TO 6 YEARS OLD CHILDREN 36
Anne-Françoise de Chambrier, Mélanie Tinnes-Vignes, Christophe Dierendonck, Nadine Giauque, Annick Fagnant, Amélie Auquière, Christophe Luxembourger, Joëlle Vlassis

TO ADD OR TO MULTIPLY? AN INTERVIEW STUDY ON PRIMARY SCHOOL CHILDREN'S PREFERENCE FOR ADDITIVE OR MULTIPLICATIVE RELATIONS 37
Tine Degrande, Lieven Verschaffel

EXAMINATION THE HORIZONTAL MATHEMATIZATION PROCESSES OF SECONDARY SCHOOL STUDENTS ACCORDING TO THE REALISTIC MATHEMATIC EDUCATION: THE EXAMPLE OF PROBABILITY SUBJECT 38
Bahar Dinçer, Süha Yılmaz

USING SELF-LEARNING GUIDE TO SUPPORT STUDENT TALK IN A LARGE-SIZE CLASS 39
Lianchun Dong

A SECOND GRADER'S SHIFTING DESCRIPTIONS OF TIME ON AN ANALOG CLOCK 40
Darrell Earnest, Alicia Gonzales

LEARNING COMPLEX SPATIAL VERBS IN A BILINGUAL MATHEMATICS CLASS 41
Cris Edmonds-Wathen

FINGER PATTERNS AS MEANS TO EXPERIENCE NUMBERS' PART-PART-WHOLE RELATIONS 42
Anna-Lena Ekdahl, Camilla Björklund, Ulla Runesson Kempe

LEVELS OF GENERALIZATION AND THE SOLO TAXONOMY 43
Rabih Raif EL Mouhayar

HOW DO PROSPECTIVE MATHEMATICS TEACHERS DEFINE THE BASIC CONCEPTS IN NUMBERS AND OPERATIONS CONTENT AREA? 44
Yusuf Emre Ercire, Serkan Narlı

TOWARDS A FRAMEWORK FOR UNDERSTANDING THE CHOICE AND USE OF EXAMPLES IN TEACHER EDUCATION MULTILINGUAL MATHEMATICS CLASSROOMS 45
Anthony A. Essien

PUZZLE-BASED LEARNING IN UNIVERSITY MATHEMATICS: STUDENTS' PERSPECTIVES 46
Tanya Evans, Sergiy Klymchuk, Mike Thomas

HOW CAN WE PROMOTE PRE-SERVICE TEACHERS' REFLECTION ON STUDENTS' MATHEMATICAL THINKING? 47
Susan Kathleen Forsythe, Fay Baldry

CHARACTERIZING TEACHER FOLLOW-UP MOVES 48
Julianne Elsie Foxworthy, Judit N Moschkovich

ERROR ANALYSIS AND METACOGNITION OF COLLEGE STUDENTS 49
Mark Lester Garcia, Maria Alva Aberin

GAP THINKING DOES NOT FULLY EXPLAIN SOME FRACTION COMPARISON DIFFICULTIES 50
David Maximiliano Gomez, Pablo Dartnell, Mabel Urrutia

PROSPECTIVE TEACHERS' STRATEGIES TO SOLVE NON-DECIMAL ADDITION PROBLEMS 51
Johanna Goral, Ulrich Kortenkamp

THE MATHCITYMAP APP – A GAMIFIED MATH TRAIL EXPERIENCE 52
Iwan Gurjanow, Matthias Ludwig

CHANGES IN WHOLE CLASS DIALOGIC ARGUMENTATION DURING FIVE MATHEMATICS LESSONS 53
Markus Hähkiöniemi, Jenna Hiltunen, Kaisa Jokiranta, Pasi Nieminen, Sami Lehesvuori, John Francisco, Jouni Viiri

INSERVICE TEACHERS' PERSPECTIVES OF PARALLELOGRAM DEFINITIONS 54
Aehsan Haj Yahya, Wajeeh Daher, Osama Swidan

WHAT IS CRITICAL IN ORDER TO LEARN THE AVERAGE RATE OF CHANGE? 55
Per Håkansson, Robert Gunnarsson

PRE-SERVICE MATHEMATICS TEACHERS' EVOKED CONCEPT IMAGES OF NUMBER SYSTEMS 56
Jani Hannula

IMPROVING GRADE 4 MULTIPLICATIVE REASONING IN SOUTH AFRICA: FINDINGS FROM AN INTERVENTION STUDY 57
Sameera Hansa, Hamsa Venkatakrishnan, Michael Askew

EXAM VS. SELF-ASSESSMENT - STUDENTS' ACHIEVEMENT GOAL ORIENTATIONS IN TWO DIFFERENT ASSESSMENT MODELS 58
Jokke Häsä, Johanna Rämö, Juuso Henrik Nieminen, Jussi Leppinen

FOSTERING CRITICAL THINKING THROUGH MATHEMATICAL PROBLEM SOLVING BASED ON THE PERSPECTIVE OF CRITICAL MATHEMATICS EDUCATION 59
Yuichiro Hattori, Takuya Baba

TEACHERS' MATHEMATICAL KNOWLEDGE OF DEFINITIONS IN GEOMETRY 60
Nader Helf, Muhammad Abu-Naja

PROMPTS, TECHNOLOGY AND PROBLEM SOLVING 61
Alexánder Hernández Hernández, Josefa Perdomo Díaz, Matías Camacho Machín

USING IMAGERY WITH GEOGEBRA TO ASSIST WITH MATRIX MULTIPLICATION: A COLLABORATIVE PROJECT 62
Dave Hewitt, Stephanie Treffert-Thomas, Barbara Jaworski, Nikolaos Vlaseros, Marinos Anastasakis

THE "TEAMS" PROFESSIONAL DEVELOPMENT PROGRAM FOR ENHANCING EXPLORATIVE INSTRUCTION 63
Einat Heyd-Metzuyanim, Talli Nachlieli, Merav Weingarden, Rinat Baor

HOW DO STUDENTS VISUALIZE FRACTIONS? 64
Stefan Hoch, Frank Reinhold, Bernhard Werner, Jürgen Richter-Gebert, Kristina Reiss

BACKWARD TRANSFER EFFECTS WHEN LEARNING ABOUT QUADRATIC FUNCTIONS 65
Charles Hohensee, Laura Willoughby, Sara Gartland

INTERACTIONS OF KNOWLEDGE AND BELIEFS IN MATHEMATICS PROFESSIONAL DEVELOPMENT 66
Jennifer Holm, Ann Kajander

DEVELOPMENT OF A SCALE FOR IDEAL MATHEMATICS TEACHING BEHAVIOR AND A COMPARISON BETWEEN TAIWAN AND MAINLAND CHINA 67
Feng-Jui Hsieh, Ting-Ying Wang, Qian Chen

TEACHERS’ USE OF EXAMPLES AND ACCOMPANYING MATHEMATICAL DISCOURSE: THE CASE OF DERIVATIVE......................... 68
Chih Hsien Huang

CHILDREN’S PERFORMANCE OF RECOGNIZING ANGLES OF INCLINATION, TURNING, AND ROTATION... 69
Hsin-Mei E. Huang, Chia-Chun Hsieh, Shu-Min Tu

PRE-SERVICE TEACHERS AS LEARNERS OF PEDAGOGICAL CONTENT KNOWLEDGE IN COMBINATORICS 70
Veronika Hubeňáková, Ingrid Semanišinová

APPROPRIATING INSCRIPTIONS AND MANIPULATIVES............................. 71
Salvador Huitzilopochtli, Judit Moschkovich

THE COMPREHENSION OF RELATIONAL CONCEPTS (<, >, =) BY PRE-SERVICE AND PRESCHOOL TEACHERS – PERSPECTIVE OF “QUANTITY” 72
Bat-Sheva Ilany, Dina Hassidov

THE MULTIPLE FUNCTIONS OF SCAFFOLDING IN SUPPORTING STUDENTS' TASK COMPLETION 73
Jenni Ingram, Nick Andrews, Andrea Pitt

MODIFYING THE TRADITIONAL PUBLIC VIEW OF MATHEMATICS 74
Hanan Innabi

HOW DO PRE-SERVICE TEACHERS APPLY THEIR KNOWLEDGE FOR TEACHING MATHEMATICS AND ECONOMICS? 75
Colin Jeschke, Hannes Saas, Christiane Kuhn, Olga Zlatkin-Troitschanskaia, Anke M. Lindmeier

RELATIONSHIPS BETWEEN MATHEMATICS TEACHER EMOTIONS AND INSTRUCTIONAL APPROACHES 76
Zheng Jiang, Ida Ah Chee Mok, Jinbo Tang

TEACHERS' AND STUDENTS' PERCEPTIONS OF THE GAP BETWEEN SECONDARY AND TERTIARY MATHEMATICS 77
Helena Johansson, Magnus Österholm, Liselott Flodén, Pia Heidtmann

PROMOTING SECONDARY STUDENTS' SHIFTS TO COVARIATIONAL REASONING: THEORY NETWORKING AND TASK DESIGN 78
Heather Lynn Johnson, Evan McClintock, Amber Gardner

PROSPECTIVE TEACHERS' LEARNING OPPORTUNITIES TO POSE MATHEMATICAL MODELING PROBLEMS 79
Hyunyi Jung, Marta Magiera

THE LANDSCAPE OF EARLY ALGEBRA TEACHING AND LEARNING 80
Thomas Kaas

OPPORTUNITY TO LEARN MATHEMATICS AFFORDED BY TWO COMPETENT TEACHERS IN THEIR LESSONS 81
Berinderjeet Kaur, Tin-Lam Toh, Wei-Yeng Karen Toh

STUDENTS' CONCEPTION OF SPANNED SPACE AND ITS RELATION TO CONCEPTION OF LINEAR INDEPENDENCE 82
Mitsuru Kawazoe

RE-VIEWING MATHEMATICAL PRACTICE 83
Nadia Stoyanova Kennedy

TEACHERS' QUERIES ABOUT ALGEBRA TEACHING 84
Cecilia Kilhamn, Ann-Sofi Röj-Lindberg

HOW DENSE ARE RATIONAL NUMBERS?: A NEW MATERIALIST APPROACH TO THE NUMBER LINE 85
Doyen Kim, Oh Nam Kwon

THE INTERNATIONAL CLASSROOM LEXICON PROJECT: IDENTIFYING AND DOCUMENTING KOREAN MIDDLE SCHOOL MATHEMATICS CLASSROOM PRACTICES 86
Hee-jeong Kim, Hyungmi Cho

REFLECTION OF A BILLIARD ACTIVITY UPON REASONING PROCESSES IN TEACHING QUADRILATERALS 87
Ayhan Kızıltoprak, Nilüfer Yavuzsoy Köse

INVESTIGATING EIGHT GRADERS' WAYS OF THINKING ASSOCIATED WITH PROBLEM SOLVING IN CONTEXT OF MATHEMATICAL LITERACY 88
Fatma Kızıltoprak, Tangül Kabael

CRITICAL INCIDENTS SHAPING TEACHERS' CONCEPTIONS OF ARGUMENTATION IN PRIMARY EDUCATION 89
Peter Michael Klöpping, Ana Kuzle

NON-ROUTINE PROBLEM POSING SKILLS OF PROSPECTIVE MATHEMATICS TEACHERS 90
Tuğçe Kozaklı Ülger, Yeliz Yazgan

ESTABLISHING REPRESENTATIONAL MEANING OF GESTURES – LEARNING FROM THE DEAF CLASSROOM 91
Christina M. Krause

INITIATING STUDENT DISCOURSES WITH SILENT VIDEO TASKS IN MATHEMATICS CLASSROOMS 92
Bjarnheiður (Bea) Kristinsdóttir, Freyja Hreinsdóttir, Zsolt Lavicza

PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' READING OF CURRICULUM MATERIALS 93
Na-Young Ku, Hyun Young Kang, InYong Choi

GERMAN PROSPECTIVE TEACHERS' UNDERSTANDING OF PLACE VALUE 94
Silke Ladel, Ulrich Kortenkamp, Johanna Goral, Eva Thanheiser

SUPPORTING QUALITY OF LEARNING IN UNIVERSITY MATHEMATICS: A COMPARISON OF TWO INSTRUCTIONAL DESIGNS 95
Juulia Josefiina Lahdenperä

STORYTELLING AS A MEANS TO COMMUNICATE MATHEMATICS AND TO ENGAGE STUDENTS EMOTIONALLY 96
Elizabeth Lake

PROOF BY INDUCTION – A COMPARATIVE STUDY 97
Niclas Larson, Kerstin Pettersson

MIDDLE SCHOOL STUDENTS' PROPORTIONAL REASONING 98
Kerstin Larsson

THE EFFECTS OF A PEDAGOGY COURSE ON PRE-SERVICE MATHEMATICS TEACHERS' BELIEFS AND CONFIDENCE ABOUT TEACHING ALGEBRA 99
Wing Fat Lau

STUDENTS' CONSOLIDATION THROUGH PROBLEM POSING AND SOLVING ACTIVITIES 100
Kyungwon Lee, Oh Nam Kwon

EXPLORING HOW A REFLECTION INSTRUCTIONAL APPROACH INFLUENCED A STUDENT'S LEARNING OF MATHEMATICAL PROBLEM SOLVING.. 101
Shin-Yi Lee

WELL-DEFINED AND ILL-DEFINED INITIAL STATE AND GOAL STATE OF TASKS AND THEIR EFFECT ON TASK DIFFICULTY 102
Matthias C. Lehner, Kristina Reiss

THE JOY OF LEARNING MATHEMATICAL MODELLING: NETWORKED LEARNING COMMUNITY AS AN ACTIVITY SYSTEM 103
Swee Ling Leong, Tan Liang Soon

RELATIONSHIPS BETWEEN TRAIT EMOTIONAL INTELLIGENCE AND DIFFERENT MATHEMATICAL SKILLS IN ELEMENTARY STUDENTS.. 104
DI LIU

COMMUNICATING BIRTHDAYS' IN PRESCHOOL A WAY FOR CHILDREN TO UNDERSTAND TIME 105
Marita Lundström

NOTICING STUDENT GENERALIZATIONS AND JUSTIFICATIONS: DOES TASK CONTEXT MATTER? 106
Marta T. Magiera, Vecihi S. Zambak

PROBLEM SOLVING IN THE EYES OF EGYPTIAN TEACHERS: A CASE STUDY AT A NATIONAL CURRICULUM SCHOOL............................. 107
Mariam Magdi Maurice Makramalla, Andreas Stylianides

STIMULUS OF PROBABILISTIC THINKING BY ENGAGING CHILDREN AND PRIMARY TEACHERS IN GAME INVENTION.................. 108
Uldarico Malaspina, Martín Malaspina

INSTRUCTOR CREATION OF LECTURE NOTES WITH DIGITAL TEXTBOOKS 109
Angeliki T Mali, Vilma M Mesa

THE ε-δ DEFINITION FOR ONE VARIABLE REAL FUNCTION REVISITED 110
Joanna Mamona-Downs, Christos Kourouniotis

STUDENTS' DIFFICULTIES WHEN SOLVING SPATIAL TASKS 111
Angel Mizzi

THE CONVERSION OF PERSPECTIVES ON ANGLES THROUGH THE OPERATION OF SIGNS AND ATTENTION 112
Sung-Jae Moon, Kyeong-Hwa Lee

THE INFLUENCE OF MATHEMATICS TEACHERS' COMPETENCIES, ATTITUDE AND INTEREST ON THEIR PROFESSIONAL BEHAVIOR 113
Achmad Mudrikah

THE ROLE OF INTERACTIVE WHITEBOARDS TO SUPPORT A GROWTH MINDSET IN ALGEBRA LEARNING 114
Margrethe Naalsund

A COMMOGNITIVE ANALYSIS OF GRADE 12 LEARNERS' PARTICIPATION IN A SITUATED MATHEMATICAL ACTIVITY 115
Williams Chapasuka Ndlovu, Willy Mwakapenda

EXPLORING PRE-SERVICE MATHEMATICS TEACHERS' SUBJECT MATTER KNOWLEDGE OF SCHOOL MATHEMATICS CONCEPTS 116
Zanele Annatoria Ndlovu

PROOF COMPREHENSION OF UNDERGRADUATE STUDENTS 117
Silke Neuhaus, Stefanie Rach

IN-SERVICE MATHEMATICS TEACHERS' VIDEO-BASED NOTICING OF 3D PRINTING PENS 'IN ACTION' 118
Oi-Lam Ng

NUMBER ESTIMATION, WORKING MEMORY AND QUANTITATIVE REASONING: A STUDY WITH 3RD AND 4TH GRADE SCHOOL CHILDREN 119
Camila Nogues, Beatriz Dorneles

HOW DOES VIDEO ANALYSIS OF TEACHING LEAD TEACHERS TO ACTION, OR TO CHANGE? 120
Jihwa Noh, Yeon Kim

INFLUENCE OF AN OUT-OF-SCHOOL MATHEMATICS MENTORING AND TUTORING PROGRAMME ON PARTICIPANTS' IDENTITY FORMATION 121
Forster Ntow

PRIMARY STUDENTS EMOTIONS TOWARDS MATHEMATICS 122
Martin Nyman

A TOOL FOR PLANNING AND ORCHESTRATING
MATHEMATICAL DISCUSSIONS.. 123
Rimma Nyman, Cecilia Kilhamn, Florenda Callos Cronberg,
Lena Knutsson, Christina Skidras, Susanne Frisk, Britt Holmberg

DIFFERENCE AND REPETITION – INSTRUCTIONAL EXAMPLES 124
Constanta Olteanu, Lucian Olteanu

TEACHERS' ABILITIES TO IDENTIFY PROPORTIONAL
SITUATIONS.. 125
Chandra Orrill, Travis Weiland, Rachael Brown

RESEARCH SITUATIONS IN ELEMENTARY SCHOOL 126
Cecile Ouvrier-Buffet

CONSTRUCTING AXES OF SYMMETRY AND VERTEX
FOR QUADRATIC FUNCTIONS THROUGH
QUANTITATIVE REASONING 127
Aytuğ Özaltun Çelik, Esra Bukova Güzel

PROOF TRIPOD: PROOF IMAGE, ENLIGHTENMENT & FEELINGS 128
Ozan Pala, Serkan Narlı, Esra Aksoy,
Yusuf Emre Ercire, Mehmet Akif Aksoy

SECOND-GENERATION MIGRANT STUDENTS FROM SINGAPORE:
A PRELIMINARY STUDY ON THEIR MATHEMATICS
PERFORMANCE IN AUSTRALIA 129
Jiahui Pan

PROSPECTIVE MATHEMATICS TEACHERS' SELF-EFFICACY
BELIEFS ON USING HISTORY IN MATHEMATICS TEACHING.................... 130
Ekaraj Pandit

STUDENTS' INTRODUCTION TO THE AXIOMATIC FOUNDATION
OF MATHEMATICS THROUGH GEOMETRY.. 131
Maria Papageorgiou, Marianna Tzekaki

UNCATEGORIZABLE CASES IN DEVELOPING A FRAME FOR
ANALYZING DIFFERENT MEANINGS OF THE CONCEPT OF THE
VARIABLE MEDIATED BY MATHEMATICAL TASKS 132
Anna-Maija Partanen, Pieti Tolvanen

USING MANIPULATIVES TO TEACH FRACTIONS IN CANADIAN
AND CHINESE ELEMENTARY SCHOOLS 133
Aihui Peng, Anthony N. Ezeife, Bo Yu, Yueqiang Shang

INTEGRATING INSTRUMENTS TO CAPTURE PRESERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS AND ITS TEACHING 134
Josefa Perdomo Díaz, Rodrigo Ulloa, María Victoria Martínez Videla, Francisco Rojas

A TROUBLE AT THE INTERFACE OF DIFFERENT DISCIPLINARY FIELDS (MATHEMATICS EDUCATION, PSYCHOLOGY AND COGNITIVE SCIENCES): DYSCALCULIA............ 135
Florence Peteers

CONCEPTIONS AND BELIEFS OF THE PROFESSORS OF NUMERICAL METHODS IN THE ENGINEERING DEGREES AT ECUADORIAN UNIVERSITIES 136
Maritza Pinta, Juan José Moreno Balcázar, Francisco Gil Cuadra, Ana Belén Montoro Medina

ASSESSMENT FOR LEARNING IN FIRST SCHOOL YEARS............ 137
Jorge Pinto, Leonor Santos

REFLEXIVE ABILITIES OF THE 9TH GRADE STUDENTS AND "MANY-VALUED" GEOMETRIC PROBLEMS 138
Natalia Podkhodova

PARENTS' ATTITUDES TOWARDS MATHEMATICS, HOME MATH-RELATED ACTIVITIES AND THEIR EFFECTS ON PRESCHOOLERS' SKILLS IN EARLY NUMERACY............ 139
Débora Poncelet, Youssef Tazouti, Ariane Baye, Nadine Giauque, Sylvie Kerger, Véronique Jans, Chantal Tièche Christinat, Mélanie Tinnes-Vigne

STUDENTS OF RELIGION STUDYING SOCIAL CONFLICTS THROUGH SIMULATION AND MODELLING - A PILOT STUDY 140
Amrit Bahadur Poudel, F. LeRon Shults, Pauline Vos

STUDENTS' EXPECTATIONS CONCERNING STUDYING MATHEMATICS AT UNIVERSITY 141
Stefanie Rach, Laura Engelmann

CONFLICT OR COMPROMISE: PRE-SERVICE MATHEMATICS TEACHERS' EXPERIENCES OF SCHOOL AND UNIVERSITY MENTORS 142
Viren Ramdhany

SHIFTS IN MATHEMATICS INTERACTIONS BETWEEN GRADE 8 STUDENTS FROM AN ENACTIVIST PERSPECTIVE............ 143
Paola Ramirez

SUPPORTING UPPER SECONDARY SCHOOL STUDENTS' INTEREST IN PLAYFUL LEARNING AND PEER TUTORING IN A NEW LEARNING ENVIRONMENT 144
Jenni Johanna Räsänen, Juha Markku Robert Oikkonen, Maija Katariina Aksela

TECHNOLOGY BASED ASSESSMENT OF EARLY NUMERACY AND LATER MATHEMATICS ACHIEVEMENT 145
Attila Rausch, Attila Pásztor

BASIC IDEAS FOR ADDITION IN PRIMARY EDUCATION TRAINEES 146
Pamela Reyes Santander, Rudolf vom Hofe

RESOLVING COGNITIVE CONFLICTS ABOUT FUNCTIONS: PRESERVICE SECONDARY MATHEMATICS TEACHERS' INTERACTIONS WITH TASKS 147
Kathryn Rhoads, Theresa A. Jorgensen, James A. Mendoza Alvarez

CHALLENGES AND OPPORTUNITIES: LISTENING TO NEWLY ARRIVED STUDENTS' MATHEMATICAL STORIES 148
Laila Riesten

APPROACHES TO MODELING BY MATHEMATICS TEACHER EDUCATORS: THE PERCEPTION OF STUDENT TEACHERS 149
Francisco Rojas, Helena Montenegro

INVESTIGATING THE STUDENT PERSPECTIVE ON SCHOOL MATHEMATICS IN THE PROCESS OF CHANGE 150
Ann-Sofi Linnea Röj-Lindberg, Kirsti Hemmi

TEACHER LEARNING IN THE CONTEXT OF LESSON STUDY: ONE TEAM, FIVE MATHEMATICS TEACHERS 151
Gerrit Roorda, Siebrich de Vries

REVISITING AND GROWING PRIOR KNOWINGS FOR LEARNERS EXPERIENCING MATHEMATICS DIFFICULTIES 152
Robyn Ruttenberg-Rozen

VALIDATING A QUESTIONNAIRE: CAPTURING THE WAY IN WHICH BELIEFS ABOUT MATH AND STUDENTS' ABILITIES INFLUENCE TEACHERS' ACTIONS IN PROBLEM SOLVING 153
Safrudiannur Safrudiannur, Benjamin Rott

CONSIDERATIONS OF CREATIVITY BY PRE-SERVICE TEACHERS OF MATHEMATICS IN THEIR MASTER'S DEGREE FINAL PROJECTS 154
Alicia Sánchez, Vicenç Font, Vahid Borji

THE TEACHER'S ROLE IN WHOLE CLASS DISCUSSION IN
DEVELOPING THE QUANTITATIVE REASONING 155
Maria de Lurdes Serrazina, Margarida Maria Rodrigues, Ana Sofia Caseiro

IS MATHEMATICS ME? CHINESE HIGH SCHOOL STUDENTS'
IDENTITY IN MATHEMATICS 156
Jie Shi, Sijie Sun, Yan Zhu, Xiaoli Lu

HOW ORDINARY LANGUAGE INFLUENCES THE FORMULATION
OF STATEMENTS WITH QUANTIFICATIONS 157
Yusuke Shinno, Tatsuya Mizoguchi, Hiroaki Hamanaka,
Takeshi Miyakawa, Susumu Kunimune

INVESTIGATING STUDENT STEM BELIEFS WITHIN A
TRANSDISCIPLINARY ROBOT BUILDING PROJECT 158
Karen Skilling

STUDENTS' AND MATHEMATICIANS' ACCEPTANCE CRITERIA
FOR MATHEMATICAL PROOFS 159
Daniel Sommerhoff, Stefan Ufer

TEACHERS' JUDGMENT ACCUCARY OF TASK DIFFICULTY
RELATED TO FUNCTIONS 160
Ute Sproesser, Markus Vogel, Tobias Dörfler, Andreas Eichler

STORYTELLING AND LEGITIMATION IN SOCIAL
JUSTICE MATHEMATICS 161
Susan Kimberley Staats, Lori Ann Laster, Kyle Whipple

TEACHERS' GROWTH: PROPORTION AND THE 5 PRACTICES 162
Olof Bjorg Steinthorsdottir, Elizabeth Hughes

"INCLUSION" AND "DISABILITY" IN THE MATHEMATICS
CLASSROOM: THE CASE OF VISUALLY IMPAIRED PUPILS 163
Angeliki Stylianidou, Elena Nardi

THE INSTRUMENTAL GENESIS IN THE DEMONSTRATION OF THE
PYTHAGOREAN THEOREM 164
Manuela Subtil, António Domingos

GENDER AS A CULTURAL CONSTRUCT: THEORETICAL ISSUES 165
Lovisa Sumpter, Guri Nortvedt

MAPPING THE DEVELOPMENT OF PROBABILISTIC
REASONING IN CHILDREN 166
Anne-Sophie Supply, Wim Van Dooren, Patrick Onghena

INVESTIGATION OF FINNISH AND GERMAN 9TH GRADE
STUDENTS' PERSONAL MEANING TO MATHEMATICS 167
Neruja Suriakumaran, Maike Vollstedt, Markku S. Hannula

EFFICENCY OF TEST-ENHANCED LEARNINING IN TEACHING NUMBER THEORY 168
Csaba Szabó, Anna Muzsnay

PRE-SERVICE KINDERGARTEN TEACHERS' EXPERIENCES ABOUT TEACHER-GUIDED MATHEMATICAL ACTIVITIES 169
Judit Szitányi, Mónika Bagota, Klára Pintér, Csaba Csíkos

DEVELOPMENT OF A COMBINED TEST ASSESSING SIXTH AND SEVENTH GRADERS' MATHEMATICAL COMPETENCE OF ALGEBRA 170
Dilek Tanışlı, Gözde Ayber, Melih Turgut

INVESTIGATING THE NATURE AND DEVELOPMENT OF A PROSPECTIVE MATHEMATICS TEACHER'S PROFESSIONAL NOTICING SKILLS DURING FIELDWORK EXPERIENCE 171
Rukiye Didem Taylan, Aysun Fendi

CONFLATING DISCOURSES: UNIVERSITY STUDENTS' EXAM RESPONSES ON INJECTIVE AND SURJECTIVE FUNCTIONS 172
Athina Thoma, Elena Nardi

WHEN IS A PATTERN NOT A PATTERN? DEVELOPING PATTERN AWARENESS IN YOUNG CHILDREN 173
Helen Rachel Thouless, Sue Gifford

DEVELOPING EARLY NUMBER COMPETENCIES THROUGH GAMES PLAYED IN SCHOOL AND AT HOME 174
Mélanie Tinnes-Vigne, Débora Poncelet, Anne-Françoise de Chambrier, Christophe Dierendonck, Youssef Tazouti, Annick Fagnant, Joëlle Vlassis

EFFICIENT AND FLEXIBLE USE OF SUBTRACTION BY ADDITION IN MULTI-DIGIT SUBTRACTION IN ELEMENTARY SCHOOL CHILDREN 175
Joke Torbeyns, Gwen Verguts, Bert De Smedt, Lieven Verschaffel

IMPROVING IN-SERVICE TEACHERS' PROBLEM POSING SKILL BY MEANS OF DIDACTIC REFLECTION 176
Carlos Torres Ninahuanca, Uldarico Victor Malaspina Jurado

SHAPE PATTERNING TASKS: THE ROLE PLAYED BY COLLABORATION IN SUPPORTING CHILDREN'S THINKING 177
Aisling Twohill

4TH GRADERS' WORKING ON A FUNCTIONAL CONTEXT: GENERALIZATION LEVELS AND INFLUENCE OF STIMULI 178
Jason Ureña, Marta Molina, Rafael Ramírez

A FRAMEWORK FOR STUDENTS' UNDERSTANDING OF INVARIANCE IN PROPORTION PROBLEMS ACROSS GRADES 179
Chaitanya Prashant Ursekar, Jayasree Subramanian

MIDDLE SCHOOL STUDENTS' REASONING PROCESSES ON SYMMETRICAL FIGURES THROUGH A DYNAMIC GEOMETRY ENVIRONMENT 180
Candas Uygan, Nilufer Yavuzsoy Kose

THE WORLD AROUND PYTHAGORAS: INTERPRETATION, LOGIC AND REASONING CONSIDERATIONS 181
Francisco Javier Vargas, Keith Stenning

TEACHERS' PERSPECTIVES TOWARD CURRICULUM 182
Linda C. Venenciano, Seanyelle L. Yagi

TEACHERS' KNOWLEDGE AND THEIR STUDENTS' MISCONCEPTIONS ABOUT THE EQUAL SIGN 183
Cornelis Vermeulen, Bronwin Meyer

USING FORMATIVE ASSESSMENT TO SUPPORT STUDENT DEVELOPMENT IN LOWER SECONDARY MATHEMATICS 184
Hanna Viitala

PRIMARY SCHOOL MATH: PROPORTIONS AS A STARTING CONTEXT? 185
Elena Vysotskaya, Iya Rekhtman, Anastasia Lobanova, Maria Yanishevskaya, Dmitry Chumachenko

LANGUAGE CHOICE AND MEANING IN PREDICTION 186
David Wagner, Karla Culligan, Joseph Dicks, Paula Kristmanson

EXPLORATORY STUDY ON THE RELATION BETWEEN THE UNDERSTANDING OF CORRELATION AND ASSOCIATION 187
Fang-Jiun Wang, Kai-Lin Yang, Xin-Yi Liu

PRESENTATION OF FRACTIONS IN PRIMARY MATHEMATICS TEXTBOOKS: THE COMBINATION OF HISTORICAL ORDER AND "TRANSCENDENT RECURSION" 188
Guangming Wang, Jian Li, Yueyuan Kang, Nan Zhang, Libao Wu, Jun Chen, Hanjun Chen

DESIGN AND EVALUATION OF A WORKSHOP TO DEVELOP TEACHERS' COMPETENCE IN DESIGNING MATHEMATICAL GROUNDING ACTIVITIES 189
Ting-Ying Wang, Kai-Lin Yang, Lan-Ting Wu, Fou-Lai Lin

THE ROLE OF REFLEXIVITY IN THE EXPRESSION OF TEACHERS' IDENTITIES IN TEACHING MATHEMATICS 190
Lise Westaway

TEACHER QUESTIONING: COMPARING MULTI-MEDIA PLATFORMS IN INITIAL TEACHER EDUCATION 191
Tracy L. Weston, Julie M. Amador

SELF-REFLECTION AS A PEDAGOGICAL TOOL FOR PRE-SERVICE ELEMENTARY TEACHERS' GROWTH IN REASONING ABOUT MEASUREMENT 192
Megan H. Wickstrom

PERSONAL CHARACTERISTICS OF TEACHERS THAT INCREASE THEIR INCLINATION TO ENACT PEDAGOGICAL PROBLEM-SOLVING ACTIVITY 193
Gaye Williams

INVESTIGATING THE RELATIONSHIP BETWEEN EPISTEMOLOGY AND GOAL-ORIENTATION IN LEARNING STATISTICS........ 194
Kai-Lin Yang, Xin-Yi Liu

EXPLORING A MATHEMATICS TEACHER'S DESIGNING OF ACTIVITIES INTEGRATING MODELING AND READING TASK........ 195
Kai-Lin Yang, Ya-Wen Lo

PROFESSIONAL DEVELOPMENT OF A MATHEMATICS TEACHER THROUGH WEB-BASED EDUCATIONAL PORTAL........ 196
Duygu Yıldırım, Nilüfer Yavuzsoy Köse, Dilek Tanışlı

DEVELOPING MATHEMATICAL THINKING POWERS: THE CASE OF USING A COMPUTER ALGEBRA SYSTEM IN DIFFERENTIAL EQUATIONS........ 197
Fereshteh Zeynivandnezhad

WHAT IS IMPORTANT IN MATHEMATICS LEARNING: PERSPECTIVE FROM THE CHINESE MAINLAND STUDENTS 198
Qiaoping Zhang

POSTER PRESENTATIONS

GEOMETRIC THINKING LEVELS AMONG COLLEGE OF EDUCATION STUDENTS 201
Muhammad Abu-Naja, Nader Helf

THE RELATIONSHIP BETWEEN COGNITIVE AND METACOGNITIVE STRATEGY FOR CONSTRUCTION OF GEOMETRY PROOF........ 202
Kosuke Akahane

"THINK, DRAG, COMMUNICATE" - A SCAFFOLDING TOOLKIT TO FOSTER ARGUMENTATION AND PROOFS 203
Giovannina Albano, Umberto Dello Iacono

SCAFFOLDED TRANSFERENCE FROM BASIC MUSICAL NOTATIONS TO BASIC FRACTIONS: A STUDY ON THE DEVELOPMENT OF PRESERVICE TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE 204
Clement Ayarebilla Ali, Hans Kweku Anderson

TWO SENSES OF UNIT WORDS AND IMPLICATIONS FOR TOPICS RELATED TO MULTIPLICATION 205
Sybilla Beckmann, Izsak Andrew

CHILDREN'S CONCEPTIONS OF TIME 206
Johannes Berggren

MATHEMATICAL LEARNING PROCESSES WITH VARYING TYPES OF MATERIAL CONDITIONING 207
Lara Billion

INTERDISCIPLINARY ACTIVITIES FOR AN INCLUSIVE MATHEMATICS EDUCATION 208
Teresa F. Blanco, Alejandro Gorgal, María Salgado, María Jesús Salinas, Pablo G. Sequeiros, Dolores Rodríguez, Cristina Núñez, José Manuel Diego

ELEMENTARY STUDENTS' INTEGER COMPARISONS 209
Laura Bofferding, Mahtob Aqazade, Sherri Farmer

A LEARNING THEORY FRAMEWORK TO GUIDE 'GROWTH MINDSET' INTERVENTIONS 210
Anita Campbell

STUDYING TEACHERS' BELIEFS ON TEACHING WHOLE NUMBERS WITH THE APPLICATION OF NINE TIMES TABLE 211
Yiming Cao, Shu Zhang

REALIZATION TREES: A COMMOGNITIVE LENS 212
Matteo Caponi, Giulia Lisarelli

MAPPING ATTENTIVENESS TO STUDENT THINKING 213
Michele Carney, Tatia Totorica, Laurie Cavey, Patrick Lowenthal

TEACHING MATHEMATICS THROUGH FINANCIAL CONTEXTS: ARE TEACHERS COMFORTABLE WITH FINANCIAL CONCEPTS? 214
Alexandre Soares Cavalcante, Annie Savard, Louis-Philippe Turineck, Azadeh Javaherpour

STATISTICS IN PRIMARY AND MIDDLE SCHOOL: HYPOTHESIS FORMULATION, DATA ANALYSIS AND CONCLUSIONS 215
Érica Cavalcanti, Gilda Lisbôa Guimarães

DO MATHEMATICS TEACHERS EXPERIENCE A PROFESSIONAL DEVELOPMENT PROGRAM IN FORMATIVE ASSESSMENT ANY DIFFERENTLY THAN OTHER TEACHERS? 216
Sara Cavka

A STUDY ON ELEMENTARY SCHOOL TEACHER TREATING STUDENT ERROR EXAMPLES IN FRACTION OPERATION 217
Chia Huang Chen

YOUNG CHILDREN CAN DISTINGUISH FAIR SITUATIONS FROM UNFAIR ONES IN PROBABILITY CONTEXT 218
Ching-Shu Chen

MATHEMATICS IDENTITY AMONG CHINESE STUDENTS 219
Donglin Chen

THE ANALYTICAL WORKING SPACE USED BY STUDENTS OF PEDAGOGY IN MATHEMATICS WHEN SOLVING A MODELLING TASK 220
Ximena Colipan, Pamela Reyes Santander, Elizabeth Montoya Delgadillo, Laurent Vivier

YOUNG CHILDREN'S VISUAL-SPATIAL THINKING IN INTERPRETING SPATIAL VOCABULARY 221
Conceição Costa, Eliane Charneca, Marisa Nascimento, José Manuel Matos

MATHEMATICS LEARNING AND APHANTASIA 222
Anita Louise Crowder

SECOND GRADERS WITH MATHEMATICAL LEARNING DISABILITIES USE MORE FINGER-COUNTING ON SMALL PROBLEMS BUT LESS ON LARGE ONES 223
Anne-Françoise de Chambrier

PROSPECTIVE TEACHERS' CONCEPTIONS OF MATHEMATICAL DEFINITIONS: ARE DEFINITIONS ARBITRARY? 224
Merve Dilberoglu, Cigdem Haser, Erdinc Cakiroglu

ON-LINE PLATFORM AS AN ARTEFACT FOR TEACHING AND LEARNING MATHEMATICS 225
António Manuel Dias Domingos, Ana Santiago, Conceição Costa, Ricardo Machado

COMPARING TEACHER QUESTIONING SEQUENCES IN CHINA AND AUSTRALIA 226
Lianchun Dong

GRADE 3/4 STUDENTS' MISCONCEPTIONS OF CUBES: A LANDSCAPE OF SOURCES OF ERRORS 227
Ann Downton, Sharyn Livy, Simone Reinhold, Susanne Wöller

CAN VISUALIZATIONS (RE)ACTIVATE CONCEPTUAL KNOWLEDGE OF CENTRAL MATHEMATICAL CONCEPTS? - AN EXPLORATORY STUDY 228
Anika Dreher, Lars Holzäpfel

THE MATHEMATICS OF TIME AND TIME MANAGEMENT 229
Darrell Earnest

AN ANALYSIS OF TEACHERS' APPRAISAL OF THE USE OF EXIT CARDS IN MATHEMATICS TEACHING........ 230
Lucy Eraikhuemen, Kate Ikponmwosa Oteze

STUDENTS' UNEXPECTED WAYS OF ENLARGING FIGURES........ 231
Kirstin Erath

CORRELATION AND REGRESSION PROBLEMS IN THE SPANISH HIGH SCHOOL TEXTBOOKS 232
María Magdalena Gea, Pedro Arteaga, Elena Molina-Portillo, María del Mar López-Martín

CAN PROBLEM FIELDS ENGAGE ALL STUDENTS IN MATHEMATICAL PROBLEM SOLVING? 233
Inga Gebel, Ana Kuzle

PROBLEM SOLVING FOR ALL: EVALUATION OF A PROBLEM SOLVING TEACHING CONCEPT FOR PRIMARY EDUCATION 234
Inga Gebel, Ana Kuzle

OVERLAPPING CIRCLES - USING PARTICIPANT GENERATED INFLUENCE MAPS AS AN INTERVIEW TOOL........ 235
Alison Mary Godfrey

STRATEGIES FOR FRACTION COMPARISON WHEN SELECTING THE LARGER OR THE SMALLER ONE........ 236
David Maximiliano Gomez, Pablo Dartnell, Mabel Urrutia

ALICANTE - BARCELONA - HELSINKI: STUDENTS' MATHEMATICAL BACKGROUND AND REQUIREMENTS TO ENTER A PRIMARY TEACHING DEGREE 237
Núria Gorgorió, Lluís Albarracín, Anu Laine, Salvador Llinares

THE IMPACT OF NUMBER-PAIRING ON STUDENTS' IDEAS ON HOW TO EVALUATE NUMERICAL EXPRESSIONS 238
Robert Gunnarsson, Ioannis Papadopoulos

PAPER CUTTING THE WORLD: FOSTERING OBJECT RECOGNITION IN GEOMETRY TEACHING THROUGH DISSECTION AND REARRANGEMENT OF SHAPES 239
Claudia-Susanne Günther

PROSPECTIVE TEACHERS´ EXPLORATION OF CINDERELLA.2,8 WITH PRIMARY CHILDREN 240
Shajahan Haja-Becker

FUNDAMENTAL TASK TO GENERATE THE IDEA OF REDUCTIO AD ABSURDUM 241
Hiroaki Hamanaka, Koji Otaki

A CASE STUDY ON FACILITATING STUDENTS' APPRECIATION OF AESTHETIC QUALITIES OF MATHEMATICAL OBJECTS 242
Hayato Hanazono

THE NOTION OF PROJECTILE MOTION – A CASE STUDY 243
Lena Hansson, Örjan Hansson, Kristina Juter, Andreas Redfors

DRUG CALCULATIONS IN NURSING EDUCATION: IS MATHEMATICS A PROBLEM? 244
Örjan Hansson, Kristina Juter

ATTUNING TO THE MATHEMATICS OF DIFFERENCE: EXAMPLES OF TEACHERS' HAPTIC CONSTRUCTIONS OF NUMBER 245
Lulu Healy, Elena Nardi, Irene Biza

QUALITY OF 1ST YEAR PRESERVICE TEACHERS' SELECTION OF TASKS, INTENDED AS A BASE FOR INTERACTION 246
Petra Hendrikse

DEVELOPING PROSPECTIVE MATHEMATICS TEACHERS' TPACK IN A DIDACTICS COURSE 247
Ana Cláudia Correia Batalha Henriques, Luis Fabián Gutiérrez-Fallas

DOING MATHEMATICS PLAYING NINE MEN'S MORRIS 248
Sabrina Héroux

DEVELOPING UNDERSTANDING OF NUMBERS IN DIFFERENT APPROACHES TO MATHEMATICS IN KINDERGARTEN 249
Dorothea Hertling, Elisabeth Rathgeb-Schnierer

ELEMENTARY TEACHERS' TAKE-AWAYS FROM SUMMER MATH INSTITUTES IN A THREE-YEAR PROFESSIONAL DEVELOPMENT PROJECT IN RURAL APPALACHIA 250
Lynn Hodge, Karen Cheng, Shande King, Nick Kim

PROSPECTIVE TEACHERS' CONCEPTIONS OF MATHEMATICS 251
Jennifer Holm

TASK DESIGN TO SCAFFOLD LOW-ACHIEVERS IN LEARNING SYSTEM OF LINEAR EQUATIONS WITH TWO UNKNOWNS 252
Hui-Yu Hsu, Ya-Wen Chen, Ying-Hao Cheng, Jian-Cheng Chen

HOW DO STUDENTS CLASSIFY THE COMMON PROPERTIES OF TWO GEOMETRIC SHAPES: AN ANALYSIS OF CHILDREN'S WRITTEN RESPONSES 253
Hui-chuan Huang, Huei-Min Wu, Ting-hua Chu

VALIDITY OF NATIONAL HIGHER EDUCATION ENTRANCE EXAMINATION MATHEMATICS TEST SCORES IN CHINA 254
Chunlian Jiang, Chuang Wang, Do-Hong Kim

UBUNTU INFUSED IN THE CONCEPTUAL UNDERSTANDING OF GEOMETRIC CONCEPTS 255
Zingiswa Mybert Jojo

AN INVESTIGATION OF COGNITIVE AND META-COGNITIVE MATHEMATICAL MODELLING COMPETENCIES OF 7TH GRADE STUDENTS 256
Ozlem Kalaycı, Ozge Gun

CODING IN MATHEMATICS EDUCATION: THE APPLICATION OF CODING ON MATHEMATICS EXPLORATION 257
Soonja Kang, Haekyung Rim, Bomi Shin, Jiwon Kim

ANALYSES OF CHANGE IN YOUNG CHILDREN'S CAPABILITY OF COMPARATIVE JUDGMENT BASED ON RECTANGLES DRAWN ON A TABLET 258
Masato Kawasaki, Taisuke Morita, Tomoko Hayashi, Shin Sugimoto, Syuuta Takenoya

LOST IN TRANSLATION: IMPLEMENTING INTERNATIONAL RESEARCH IN SWEDISH MATHEMATICS EDUCATION 259
Cecilia Kilhamn

TEACHERS' VIEWS ABOUT CHARACTERISTICS OF PROFESSIONAL DEVELOPMENT ACTIVITIES 260
Sebastian Kuntze, Jens Krummenauer

COMPARING CRITICAL ASPECTS IN LESSONS OF EXPERIENCED TEACHERS WITH VARIATION PEDAGOGY 261
Arthur Man Sang Lee, Ka Lok Wong, Ming Fai Pang, Wing Wah Ki

LEARNING MATHEMATICAL PROBLEM SOLVING BY DOING 262
Shin-Yi Lee

VISIBLE AND INVISIBLE AFFORDANCES OF A MATHEMATICS TEXTBOOK 263
Moneoang Leshota

INTERPRETATION AND CONSTRUCTION OF BAR CHARTS IN THE ADULT EDUCATION 264
Izauriana Borges Lima, Ana Coelho Vieira Selva

THE CONSTRUCTION OF THE MEASURE OF MATHEMATICAL ARGUMENTATION 265
Pi-Jen Lin, Chun-Yung Cheng, Bao-Quei Tsai

THE INVESTIGATION OF READING AND COGNITIVE COMPONENTS OF ITEM DIFFICULTIES FOR MATHEMATICAL LITERACY TEST 266
Su-Wei Lin

MATHEMATICS TEACHERS' IDENTITY WORK IN THE CONTEXT OF LESSON STUDY 267
Ana Leticia Losano, Dario Fiorentini

AN EXAMINATION OF STUDY MOTIVATION CLASSIFYING HIGH- AND LOW-PERFORMING STUDENTS IN PISA 2012 MATHEMATICAL LITERACY STUDY 268
Soi-kei Mak, Kwok-cheung Cheung, Pou-seong Sit, Man-kai Ieong

MATHEMATICS TEACHERS' EDUCATION WITHIN A TECHNOLOGY ENHANCED LEARNING ENVIRONMENT IN PROJECT FTE-LAB 269
João Filipe Matos, Neuza Pedro, Ana Pedro

THE EFFECT OF EXPOSING STUDENTS TO MATHEMATICS-NEWS-SNAPSHOTS ON THEIR IMAGE OF MATHEMATICS 270
Nitsa Movshovitz-Hadar, Atara Shriki, Ruti Segal

A COMPARATIVE ANALYSIS OF MATHEMATICS CURRICULUM BETWEEN JAPAN AND THE UNITED STATES 271
Shogo Murata

WHAT IT MEANS TO BE INTRODUCED TO MATHEMATICS: AN EXPOSITION OF SECONDARY SCHOOL STUDENTS FROM AN INTERVENTION STUDY 272
Williams Chapasuka Ndlovu, Willy Mwakapenda

NUMBER ESTIMATION ABILITY IN 3RD AND 4TH GRADE SCHOOL CHILDREN 273
Camila Nogues, Beatriz Dorneles

GENDER AS A CULTURAL CONSTRUCT: METHODOLOGICAL ISSUES IN QUESTIONNAIRE DEVELOPMENT 274
Guri Nortvedt, Lovisa Sumpter

ARTISANS, MACHINES AND INFORMAL LEARNING: THE CASE OF FIBRE MATHEMATICS 275
Katherine Claire O'Brien

EPISTEMIC NETWORK ANALYSIS AS A LENS TO UNDERSTAND TEACHER KNOWLEDGE OF PROPORTIONS 276
Chandra Orrill, Rachael Brown

GRADUATE STUDENTS' CONCEPTUAL UNDERSTANDING OF SHIFTING BETWEEN REPRESENTATIONS 277
Tikva Ovadiya, Ruti Segal

GENERALIZING THE IDEAS REGARDING TRANSLATION OF A PARABOLA BY A MATHEMATICAL MODELING ACTIVITY 278
Aytuğ Özaltun Çelik, Esra Bukova Güzel

THE EFFECT OF COMMUNICATIVE MULTI-MEDIA MATERIALS OF MATHEMATICS WORD PROBLEM 279
Yun Hsia Pai

SPANISH PRIMARY MATHEMATICS PROSPECTIVE TEACHERS' GENERALIZATIONS IN FUNCTIONAL THINKING 280
Irene Polo-Blanco, Ana Cláudia Correia Batalha Henriques, Hélia Margarida Oliveira

VISUAL REPRESENTATIONS OF MULTIPLICATIVE RELATIONSHIPS BY STUDENTS LEARNING MATHEMATICS 281
Elena Polotskaia, Alexandre Cavalcante, Annie Savard

THE SOCIAL PROCESS OF STUDENTS' COLLABORATIVE PROBLEM SOLVING IN CHINA 282
Xuefeng Qiao, Yiming Cao

BIOGRAPHY, EMOTION AND MOTIVATION IN MATHEMATICS STUDIES: DESIGN OF A COURSE FOR STUDENT TEACHERS 283
Karen Reitz-Koncebovski, Johanna Goral, Ulrich Kortenkamp

PROBLEM SOLVING TO DIFFERENTIATE GRADES OF MATHEMATICAL TALENT 284
Juan Miguel Ribera, Adela Jaime, Rafael Ramirez

SIX YEARS OLD PUPILS' KNOWLEDGE ABOUT RECTANGLES 285
Maria Paula Pereira Rodrigues, Lurdes Serrazina

DIAGNOSIS OF BELIEFS AND KNOWLEDGE THAT PRE-SERVICE PRIMARY TEACHERS HAVE ABOUT SCHOOL MATHEMATICS, ITS LEARNING AND TEACHING 286
Francisco Rojas, María Victoria Martínez, Eugenio Chandia, Andrés Ortiz, Josefa Perdomo-Díaz, Cristian Reyes, Rodrigo Ulloa

EMBODIED FRACTIONS: CONCEPTUAL DIFFICULTIES IN THE LIGHT OF GROUNDING METAPHORS 287
Anna Ida Säfström, Caroline Nagy

CHANGING PARADIGMS IN PROBLEM SOLVING: AN EXAMPLE OF A PROFESSIONAL DEVELOPMENT WITH ELEMENTARY SCHOOL TEACHERS 288
Annie Savard, Alexandre Cavalcante, Elena Polotskaia

SUPPORTING MIDDLE-SCHOOL STUDENTS' SPATIAL ORIENTATION BY USING ORIENTEERING ACTIVITIES IN REAL-WORLD ENVIRONMENT 289
H. Seda Sezgin, Tuba Ada

SELF-RESPONSIBILITY IN MATHEMATICS LEARNING OF MACAO STUDENTS IN PISA 2012 MATHEMATICS STUDY 290
Pou Seong Sit, Soi-kei Mak, Kwok-cheung Cheung, Man-kai Ieong

TEACHERS PROMOTING ACTIVE STUDENT DIALOGUE THROUGH LISTENING AND QUESTIONING ACTIVITIES 291
Marie Sjöblom

CAN VALIDATING PROOFS HELP TO CONSTRUCT PROOFS? 292
Daniel Sommerhoff

SENSUOUS SYNTAX: COLORING CONJECTURES WITH SOUND, SILENCE, GESTURE AND GRAMMAR 293
Susan Kimberley Staats

SUBJECT MATTER ANALYSIS: THE CASE OF CRYPTOGRAPHY WITH ELLIPTIC CURVES 294
Hannes Stoppel, Benjamin Rott

THE DANGER OF PICKING THE WRONG DRESS 295
Csaba Szabó

EFFICIENCY OF TEST-ENHANCED LEARNING IN TEACHING ELEMENTARY GEOMETRY 296
Csaba Szabó, Janka Szeibert

POSITIONING OF MATHEMATICS WITHIN INTEGRATED STEM EDUCATION: THE VIEWPOINT OF LEARNING MATHEMATICS 297
Ayumu Takeuchi

HOW WOULD STUDENTS DECIDE IF A LINE IS STRAIGHT? 298
Hak Ping Tam, Wei-Min Sun, Shu-Chi Hsieh

PRIMARY SCHOOL TEACHERS' ENJOYMENT (OR NOT!) OF MATHEMATICS AND ITS TEACHING 299
Eleni Tsikalaki, Christina Misailidou

GROUNDING THE BIG IDEA OF FACTORING QUADRATIC TRINOMIALS WITHIN A LEARNING ACTIVITY 300
Tai-Yih Tso, Feng-Lin Lu

INVESTIGATING THE MATHEMATICS TEACHING EFFICACY BELIEFS OF STUDENT TEACHERS 301
Aisling Twohill, Siún NicMhuirí, Lorraine Harbison

STUDENTS' REASONING IN THE CLASSROOM: AN APPROACH FOR ANALYSIS 302
Hendrik Van Steenbrugge, Mathias Norqvist

VALIDATING A METHODOLOGICAL INSTRUMENT TO FOSTER CREATIVITY IN SOLVING OPEN-ENDED TASKS IN MATHEMATICS TEXTBOOKS 303
Yuly Vanegas, Javier Diez-Palomar, Joaquim Giménez

MATHEMATICS IN THE SWEDISH FRITIDSHEM CURRICULUM: A POLICY ENACTMENT PERSPECTIVE 304
Anna Wallin, Eva Norén, Paola Valero

CULTIVATING MATHEMATICS PRACTICE IN A COMPUTATIONAL THINKING INFUSED ROBOTIC ACTIVITY 305
Sasha Wang, Dazhi Yang, Steve Swanson, Yu-hui Ching, Younghyun Baek, Bharskar Chittoori

ACTIVITY FOR SPATIAL REASONING DEVELOPMENT OF ELEMENTARY SCHOOL CHILDREN 306
Satoshi Watanabe

DEVELOPMENT OF A MEASUREMENT ESTIMATION TEST FOR LENGTH, AREA, AND VOLUME 307
Dana Farina Weiher

PLAYING A COOPERATIVE GEOMETRY BOARD GAME: WHO BENEFIT? WHEN AND HOW? 308
Huei-min Wu, Hui-chuan Huang, Ting-hua Chu

THE EFFECT OF EXPERTISE LEVEL IN GEOGEBRA ON TECHNOLOGICAL AND PEDAGOGICAL AND CONTENT KNOWLEDGE AMONG IRANIAN SECONDARY MATHEMATICS TEACHERS 309
Fereshteh Zeynivandnezhad, Nasrin Asgharzadeh, Maryam Kazemi

TEACHERS' PERCEPTIONS AND ENACTMENT OF THE MATHEMATICAL CURRICULUM REFORM IN CHINA 310
Wenjun Zhao, Ida Ah Chee Mok

INDEX

INDEX OF AUTHORS AND CO-AUTHORS (VOLUME 5) 313

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

HISTORY OF PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution. It is an official sub-group of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

Efraim Fischbein, Israel	Stephen Lerman, UK
Richard R. Skemp, UK	Gilah Leder, Australia
Gerard Vergnaud, France	Rina Hershkowitz, Israel
Kevin F. Collis, Australia	Chris Breen, South Africa
Pearla Nesher, Israel	Fou-Lai Lin, Taiwan
Nicolas Balacheff, France	João Filipe Matos, Portugal
Kathleen Hart, UK	Barbara Jaworski, UK
Carolyn Kieran, Canada	

The current president is Peter Liljedahl, Canada.

THE CONSTITUTION OF PME

The constitution of PME was adopted by the Annual General Meeting on August 17, 1980 and changed by the Annual General Meetings on July 24, 1987, on August 10, 1992, on August 2, 1994, on July 18, 1997, on July 14, 2005 and on July 21, 2012.

The major goals of the group are:

- to promote international contact and exchange of scientific information in the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

All information concerning PME and its constitution can be found at the PME website: www.igpme.org

PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with the aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other during working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued three times a year, and can be found on the PME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

WEBSITE OF PME

All information concerning PME, its constitution, and past conferences can be found at the PME website: www.igpme.org

HONORARY MEMBERS OF PME

Efraim Fischbein (Deceased)

Hans Freudenthal (Deceased)

Joop Van Dormolen (Retired)

PME ADMINISTRATIVE MANAGER

The administration of PME is coordinated by the Administrative Manager:

Bettina Rösken-Winter

Humboldt Universität, Berlin

Postal address: Unter den Linden 6
D-10099 Berlin, Germany

Phone: +49 (0) 30 2093-5857

Email: info@igpme.org

INTERNATIONAL COMMITTEE OF PME

Members of the International Committee (IC) are elected for four years. Every year, four members retire and four new members are elected. The IC is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made at the Annual General Meeting (AGM) during the conference.

The IC work is led by the PME president who is elected by PME members for three years.

PROCEEDINGS OF PREVIOUS PME CONFERENCES

The table includes the ERIC numbers, links to download, ISBN/ISSN of the proceedings, and/or the website address of annual PME.

No.	Year	Location	ERIC number, ISBN/ISSN and/or website address
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945, ISBN 3-922211-00-3
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943, ISBN 2-87092-000-8
7	1983	Shoresh, Israel	ED241295, ISBN 965-281-000-2
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, The Netherlands	ED411130 (vol. 1) ED411131 (vol. 2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532, ISSN 0771-100X
12	1988	Veszprém, Hungary	ED411128 (vol. 1) ED411129 (vol. 2)
13	1989	Paris, France	ED411140 (vol. 1) ED411141 (vol. 2) ED411142 (vol. 3)
14	1990	Oaxtepex, Mexico	ED411137 (vol. 1) ED411138 (vol. 2) ED411139 (vol. 3)
15	1991	Assisi, Italy	ED413162 (vol. 1) ED413163 (vol. 2) ED413164 (vol. 3)
16	1992	Durham, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537

19	1995	Recife, Brazil	ED411134 (vol. 1) ED411135 (vol. 2) ED411136 (vol. 3)
20	1996	Valencia, Spain	ED453070 (vol. 1) ED453071 (vol. 2) ED453072 (vol. 3) ED453073 (vol. 4) ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol. 1) ED416083 (vol. 2) ED416084 (vol. 3) ED416085 (vol. 4)
22	1998	Stellenbosch, South Africa	ED427969 (vol. 1) ED427970 (vol. 2) ED427971 (vol. 3) ED427972 (vol. 4) ISSN 0771-100X
23	1999	Haifa, Israel	ED436403, ISSN 0771-100X
24	2000	Hiroshima, Japan	ED452301 (vol. 1) ED452302 (vol. 2) ED452303 (vol. 3) ED452304 (vol. 4) ISSN 0771-100X
25	2001	Utrecht, The Netherlands	ED466950, ISBN 90-74684-16-5
26	2002	Norwich, United Kingdom	ED476065, ISBN 0-9539983-6-3
27	2003	Honolulu, Hawai'i, USA	ED500857 (vol.1) ED500859 (vol.2) ED500858 (vol.3) ED500860 (vol.4) ISSN 0771-100X http://www.hawaii.edu/pme27
28	2004	Bergen, Norway	ED489178 (vol.1) ED489632 (vol.2) ED489538 (vol.3) ED489597 (vol.4) ISSN 0771-100X www.emis.de/proceedings/PME28

29	2005	Melbourne, Australia	ED496845 (vol. 1) ED496859 (vol. 2) ED496848 (vol. 3) ED496851 (vol. 4) ISSN 0771-100X
30	2006	Prague, Czech Republic	ED496931 (vol. 1) ED496932 (vol. 2) ED496933 (vol. 3) ED496934 (vol. 4) ED496939 (vol. 5) ISSN 0771-100X http://class.pedf.cuni.cz/pme30
31	2007	Seoul, Korea	ED499419 (vol. 1) ED499417 (vol. 2) ED499416 (vol. 3) ED499418 (vol. 4) ISSN 0771-100X
32	2008	Morelia, Mexico	ISBN 978-968-9020-06-6 ISSN 0771-100X http://www.pme32-na30.org.mx
33	2009	Thessaloniki, Greece	ISBN 978-960-243-652-3 ISSN 0771-100X
34	2010	Belo Horizonte, Brazil	ISSN 0771-100X http://pme34.lcc.ufmg.br
35	2011	Ankara, Turkey	ISBN 978-975-429-262-6 ISSN 0771-100X http://www.arber.com.tr/pme35.org
36	2012	Taipei, Taiwan	ISSN 0771-100X http://tame.tw/pme36
37	2013	Kiel, Germany	ISBN 978-3-89088-287-1 ISSN 0771-100X http:// http://www.pme2013.de/
38	2014	Vancouver, Canada	ISBN 978-0-86491-360-9 ISSN 0771-100X http://www.pme38.com/
39	2015	Hobart, Australia	ISBN 978-1-86295-829-6 ISSN 0771-100X http://www.pme39.com

40	2016	Szeged, Hungary	ISSN 0771-100 http://pme40.hu
41	2017	Singapore	ISBN 978-981-11-3742-6 http://math.nie.edu.sg/pme41

Copies of some previous PME Conference Proceedings are still available for sale. Please contact the PME Administrative Manager (e-mail: info@igpme.org).

Members of PME can reach most of the previous proceedings books at the IGPME website (http://igpme.org). Abstracts from some articles can be inspected on the ERIC website (http://www.eric.ed.gov) and are listed in the Mathematics Education Database – MathEduc (http://www.zentralblatt-math.org/matheduc).

THE PME 42 CONFERENCE

Two committees are responsible for the organization of the PME 42 Conference: the International Program Committee (IPC) and the Local Organizing Committee (LOC).

THE INTERNATIONAL PROGRAM COMMITTEE (IPC)

Ewa Bergqvist	Umeå University (Sweden) Co-chair, LOC representative
Magnus Österholm	Umeå University (Sweden) Co-chair, LOC representative
Carina Granberg	Umeå University (Sweden) LOC representative
Lovisa Sumpter	Stockholm University (Sweden) LOC and PME representative
Peter Liljedahl	Simon Fraser University (Canada) PME President
Laurinda Brown	University of Bristol (United Kingdom) PME representative
Stanislaw Schukajlow-Wasjutinski	University of Münster (Germany) PME representative
Hamsa Venkatakrishnan	University of the Witwatersrand (South Africa) PME representative

THE LOCAL ORGANISING COMMITTEE (LOC)

Ewa Bergqvist, Tomas Bergqvist, Carina Granberg, Olof Johansson, Johan Lithner, Mathias Norqvist, Catarina Rudälv, Lotta Vingsle, and Magnus Österholm, from Umeå University, Sweden, and Lovisa Sumpter from Stockholm University, Sweden.

In addition, Umeå Congress (http://umea-congress.se) is the congress bureau involved in the organizing of PME 42.

HOSTING INSTITUTE OF PME 42

PME 42 is hosted by Umeå Mathematics Education Research Centre (UMERC) at Umeå University, Sweden.

http://www.umerc.umu.se

http://www.umu.se/english

REVIEW PROCESS OF PME 42

RESEARCH REPORTS (RR)

Research Reports are intended to present empirical or theoretical research results on a topic that relates to the major goals of PME. Reports should state what is new in the research, how the study builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The number of submitted RR proposals was 416, and 191 of them were accepted. Of those not accepted as RR proposals, 162 were invited to be re-submitted as Oral Communication (OC) and 55 as Poster Presentation (PP).

ORAL COMMUNICATIONS (OC)

Oral Communications are intended to present smaller studies and research that is best communicated by means of a shorter oral presentation instead of a full Research Report. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted OC proposals was 217, and 142 of them were accepted. Of those not accepted as OC proposals, 66 were invited to be re-submitted as Poster Presentation (PP). In the end, considering re-submissions of Research Reports as Oral Communications, 217 OCs were accepted for presentation at PME 42.

POSTER PRESENTATIONS (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than an oral presentation. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted PP proposals was 93, and 71 of them were accepted. In the end, considering re-submissions of Research Reports and Oral Communications as Poster Presentations, 129 PPs were accepted for presentation at PME 42.

COLLOQUIA (CO)

The goal of a Colloquium is to provide the opportunity to present a set of three papers that are interrelated in a particular way (e.g., they are connected through related or contrasting theoretical stances, use identical instruments or methods, or focus on closely related research questions), and to initiate a discussion with the audience on the interrelated set.

The number of submitted CO proposals was 4, and 2 of them were accepted.

RESEARCH FORUMS (RF)

The goal of a Research Forum is to create dialogue and discussion by offering PME members more elaborate presentations, reactions, and discussions on topics on which substantial research has been undertaken in the last 5-10 years and which continue to hold the active interest of a large subgroup of PME. A Research Forum is not supposed to be a collection of presentations but instead is meant to convey an overview of an area of research and its main current questions, thus highlighting contemporary debates and perspectives in the field.

The number of submitted RF proposals was 4, and all of them were accepted. However, one of the four was later withdrawn, resulting in 3 Research Forums.

WORKING GROUPS (WG)

The aim of Working Group is that PME participants are offered the opportunity to engage in exchange or to collaborate in respect to a common research topic (e.g., start a joint research activity, share research experiences, continue or engage in academic discourse). A Working Group may deal with emerging topics (in the sense of newly developing) as well as topics that are not new but possibly subject to changes. It must provide opportunities for contributions of the participants that are aligned with a clear goal (e.g. share materials, work collaboratively on texts, and discuss well-specified questions). A Working Group is not supposed to be a collection of individual research presentations (see Colloquium format), but instead is meant to build a coherent opportunity to work on a common research topic. In contrast to the Research Forum format that is meant to present the state of the art of established research topics, Working Groups are considered to involve fields where research topics are evolving.

The number of submitted WG proposals was 14. All of them were accepted, together with 1 contribution that was originally submitted as a Seminar, making a total of 15 Working Groups.

SEMINARS (SE)

The goal of a Seminar is the professional development of PME participants, especially new researchers and/or first comers, in different topics related to scientific PME activities. This encompasses, for example, aspects like research methods, academic writing, or reviewing. A Seminar is not intended to be only a presentation but should involve the participants actively.

The number of submitted SE proposals was 2, and 1 was changed and accepted as a Working Group, and 1 was accepted as a Seminar.

LIST OF PME 42 REVIEWERS

The International Program Committee of PME 42 thanks the following people for their help in the review process.

A

Adler, Jill (South Africa)
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Hegedus, Stephen (United States)
Heinze, Aiso (Germany)
Henriques, Ana Cláudia Correia Batalha (Portugal)
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Holzäpfel, Lars (Germany)
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Hospesova, Alena (Czech Republic)
Huang, Chih Hsien (Taiwan)
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Kaasila, Raimo (Finland)
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Karsenty, Ronnie (Israel)
Kaur, Berinderjeet (Singapore)
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Komatsu, Kotaro (Japan)
Kortenkamp, Ulrich (Germany)
Kouropatov, Anatoli (Israel)
Koyama, Masataka (Japan)
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Krause, Christina (Germany)
Krzywacki, Heidi (Finland)
Kullberg, Angelika (Sweden)
Kuntze, Sebastian (Germany)
Kuzle, Ana (Germany)
Kwon, Oh Nam (South Korea)
Kynigos, Chronis (Greece)

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Leikin, Roza (Israel)
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PLENARY LECTURES

SYSTEMS PERSPECTIVES ON MATHEMATICS TEACHERS' BELIEFS: ILLUSTRATIONS FROM BELIEFS ABOUT STUDENTS

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In this paper I draw on the example of one secondary mathematics teacher, and a group of such teachers, to illustrate some possible affordances of complexity science ideas along with those of belief systems for understanding and possibly influencing the belief systems of individual teachers as well as those of groups of teachers. I focus particularly on teachers' beliefs about their students, drawing attention to the ways in which these beliefs might relate to other influential beliefs that teachers hold and how they constrain the ways in which students experience mathematics and hence the opportunities that they have to learn and enjoy the discipline.

In 2006, Leder and Forgasz reviewed the previous 30 years of PME research on mathematics related affect and noted a shift toward studying affective systems. The idea that beliefs exist in systems dates back to seminal work of Green (1971) and Rokeach (1960). It underpins our recognition that it is not possible to trace linear connections between specific beliefs and particular teaching practices but rather, the influence of beliefs on practice is subtle although powerful. Roesken, Pepin, and Törner (2011) located the origins of interest in teacher beliefs in the apparent failure of the problem-solving movement. They cited work that implicates teachers' beliefs about the nature of mathematics, what it means to do mathematics, and students' apparent lack of ability in this 'failure'. In this paper, I build upon ideas initially presented in Beswick (2006) to consider ways in which ideas from complexity science (e.g., Davis & Sumara, 2005) combined with established notions of belief systems (Green, 1971), might offer ways of thinking about how mathematics teachers' beliefs can be understood and influenced, with the ultimate goal of improving students' experience and learning of mathematics. I consider the beliefs of a group of secondary mathematics teachers, focussing on one in particular, and also draw upon the role of emotions in forming and evoking beliefs.

I take an individual's beliefs to include anything that the individual regards as true and hence to encompass knowledge, thereby positioning them as primarily, if not entirely, cognitive (Philipp, 2007). Although it is often convenient to consider cognition (e.g., beliefs as defined here) and emotion separately neither can be fully understood in isolation (Hannula, 2002). Although they exist at opposite ends of McLeod's (1989) spectrum of affects, they are intertwined as is illustrated by DeBellis and Goldin's (2006) description of the interaction of meta-affect (feelings about feelings) and cognition. Hannula (2002) described emotions as biasing attention and memory (beliefs about past experiences) and thereby influencing cognition.

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BELIEF SYSTEMS

Green's (1971) metaphorical description of belief systems continues to be widely used in studies of teachers' beliefs (e.g., Beswick, 2005). Green (1971) described more strongly held beliefs as more central. The more central a belief, the more resistant it is to questioning and change. Rokeach (1960) linked the centrality of a belief to the extent of its connection to other beliefs. The resistance of centrally held beliefs to change arises from the extent of their connections and, because belief systems are dynamic, the relative centrality of beliefs varies with context (Green, 1971). Beliefs can be primary or derivative where each derivative belief is a logical (to the individual concerned) consequence of another belief (Green, 1971). Beliefs can be either evidentially or non-evidentially held. According to Green (1971), beliefs in the latter category are held for reasons such as the authority of the source, or because they support existing, centrally held beliefs. Clusters of beliefs can exist in relative isolation from one another and are likely to develop when beliefs arise in independent contexts (Green, 1971). An individual can be unaware of contradictions between beliefs held in distinct clusters, allowing them to hold conflicting beliefs without any sense of conflict (Green, 1971). Non-evidentially held beliefs are more likely than evidentially held beliefs to be held in relative isolation, and are, by definition, impervious to change even in the light of contradictory evidence (Green 1971). Implicit in this is the crucial notion, that in order to change an individual's beliefs, one must challenge the bases of those beliefs.

Pajares (1992) described a "primacy effect" arising from the fact that beliefs are formed as we interact with the world and make inferences about ourselves and the world. Many, but not all, early beliefs are revised as contradictory evidence is encountered or better explanations present (this is learning). This applies particularly to beliefs about oneself, perhaps as a student of mathematics. Existing beliefs bias our attention and influence the formation of subsequent beliefs. In addition, by having been part of an individual's belief system for a long time, beliefs formed early in life are likely to be deeply enmeshed in the individual's belief system and, therefore, to be among the most centrally held. Memories are essentially beliefs about past experience, the most enduring of which tend to be associated with a strong emotional response at the time of the experience (Spiro, 1982, cited in Nespor, 1987). This emotional response can be evoked when the memory is recalled.

Beliefs research has focussed on the beliefs of individuals. Cumulative results from collections of teachers or students who respond to a survey of some kind can be used to make general statements about teachers or students, but this is not the same as considering the shared beliefs that appear to exist among specific groups in particular locations at a given time. This is in spite of the fact that much of the work of teaching is conducted with groups of students in classes and that professional learning (PL) activities optimally involve groups of teachers working together (Schleicher, 2016). Systems approaches offer a way to think about the beliefs of groups of teachers as well as those of individual teachers (e.g., Beswick, Watson, & DeGeest, 2010).

COMPLEX SYSTEMS

The notion of complex systems arose from the recognition that traditional ways of studying the world were inadequate for certain phenomena (Davis & Simmt, 2003). Whereas Newtonian mechanics provides powerful descriptions of simple systems involving interactions between very small numbers of objects, increasing the numbers of interacting entities results in systems that are complicated. Complex systems are not merely complicated. Rather, they typically involve collections of living and hence (at some level) autonomous agents from whose interactions emerge characteristics of the collective that are not attributable to the actions of any particular individuals in the collective. Complex systems, therefore, have properties that transcend those of the individual agents that give rise to the system (Davis & Simmt, 2003).

Complex systems are often nested. For example, cells, organs, individuals, social groups, and society more broadly can all be considered to be complex systems which emerge from the interactions of agents which comprise the entities of the previous level (Davis & Simmt, 2003). Complex systems are self-organising, self-maintaining, self-renewing, and structurally determined (Davis, 2004). The latter refers to the fact that a complex system's reaction to a stimulus is related to the history, or prior learning, of the system rather than being determined entirely by the influence upon it. Because of this, the system's reaction to the same stimulus on different occasions is unpredictable. According to Miller, McDaniel, Crabtree, and Stange (2001) multiple feedback loops and nonlinear interactions among agents are the means by which complex systems engage in sense-making and improvisation in response to environmental changes, and self-organise and adapt to unpredictable events.

Various researchers have suggested interdependent and necessary but not sufficient conditions for the emergence of complexity from the interactions of a collective (e.g., Davis 2004; Seel, 2003). The five conditions that Davis and colleagues have argued apply to all complex systems are: *Diversity*, which refers to variations among the agents in a collective that provide possibilities for novel responses; *Redundancy*, which refers to the common ground among the agents of a system that enables them to interact meaningfully. It counterbalances diversity and also contributes to the robustness of the system by ensuring that the various agents within it are able to compensate for one another's weaknesses; *Enabling constraints*, that organise and focus activity while allowing the diversity present to be expressed. For complexity to emerge, the actions of individual agents cannot be completely random; *Decentralised control* that emphasises that complexity cannot be planned nor can its outcomes entirely predicted, but rather that complexity emerges from the interactions of autonomous agents; and *Neighbour interactions* between agents which, in educational settings refer most importantly to interactions between ideas. To this, Seel (2003) would add that the quality and frequency of interactions, need to be sufficient and generally higher when an organisation is in flux.

Seel (2003) also identified the need for those wanting change in an organisation to exercise *Watchful anticipation*, that is, to allow the time needed for change to emerge.

In addition, he included *Intentionality*, which he acknowledged is, itself, an emergent property of a complex system in his list of the features of such systems. Intentionality recognises that although the emergent features of a complex system cannot be controlled or fully predicted they can be influenced. Similarly, Davis (2005) described the teacher as the "consciousness of the collective" in that he/she directs the attention of the class, and selects from among possible courses of action and interpretations of events, much as the conscious mind directs the attention of an individual. The teacher's intentions are co-created by his/her own actions and those of individual class members who may have their own, perhaps quite different intentions. As illustrated by Beswick et al. (2010), a leader, in this case of a group of teachers, can, as Kurtz and Snowden (2003) explained, intervene to reinforce and stabilise patterns that appear to be emerging from the collective and that align with his/her intentions, or to structure the environment so that certain patterns of interaction are more likely and desired outcomes are therefore more likely to emerge.

BELIEF SYSTEMS AS COMPLEX

The defining characteristics of complex systems are emergence and adaptation (Davis & Sumara, 2005). Individual beliefs can be thought of as the agents that comprise belief systems. In complex systems, agents are autonomous in the sense that each is capable of independent action, but this does not necessitate them being conscious. An individual's beliefs are in some sense separate, but in terms of their implications for attitudes and behaviours they do not operate alone but rather as parts of systems (Green, 1971) from which attitudes and actions can be considered to emerge. In addition, because the relative centrality of beliefs in the system is dynamic, emergent attitudes and actions are not necessarily predictable.

Emergent phenomena in belief systems can also include beliefs that arise from the interactions of other beliefs. An illustration is provided by Schuck (1999), who found that although many prospective teachers had firm beliefs about the importance of making mathematics enjoyable, they did not believe that their own mathematical knowledge was necessarily important to their ability to teach mathematics well. Many believed the converse to be true (Schuck, 1999). The belief that mathematical ability is not necessary for effective mathematics teaching allowed them to maintain belief in themselves as potentially effective teachers (Schuck, 1999). Schuck's (1999) explanation can be understood as an unconscious act on the part of the prospective teachers, to isolate their beliefs about mathematics from their beliefs about mathematics teaching in separate clusters (Green, 1971). In this way they were able to protect from challenge their centrally held beliefs that related to their identities as teachers. In this example, the reconciling belief that 'the teacher's mathematical knowledge is not related to their ability to teach it well' emerged as a response to a disruption to the individual's beliefs about him/her self as a prospective teacher.

Teachers who hold such beliefs are likely to focus on any evidence that supports them, at least in part because of the emotion likely to be attached to the need to protect their belief in their competence as teachers of mathematics. The example also illustrates how belief systems embody their histories: The experiences which have contributed to structuring the individual's belief system are encoded in the system as beliefs (memories with their emotional entailments) that influence the way in which the system responds to subsequent events, interpreting some as evidence of a belief that is now important to the individual's identity. It illustrates how belief systems are adaptive and hence learning, and, at least to the extent that interactions among beliefs and shifts in relative importance or centrality are unconscious, are self-organising. One can also imagine sub-systems of beliefs nested within wider systems; for example, beliefs about teaching a particular mathematics topic nested within beliefs about mathematics teaching more generally.

Ideas that are central to complexity theory have the potential to enrich understandings of the nature of belief systems as described by Green (1971). The fact that biological systems tend to be complex reminds us that the structure of belief systems is best thought of in biological as opposed to architectural terms (Towers & Davis, 2002). The structure of an organism implies the evolutionary result of chance as opposed to deliberate planning, and ongoing adaptation and change as opposed to rigidity and permanence. That is, the structure of a complex system (of beliefs) is seen as in a constant state of flux as component beliefs interact and the system itself interacts with and adapts to its environment (Miller et al., 2001). The system's structure embodies its evolutionary history and is characterised by emergence and continuity. Nevertheless, the system can arrive at quite stable states in which predictable patterns of interaction can ossify in the absence of new inputs.

Conditions for complex emergence and belief systems

In terms of Davis' (2004), conditions for complex emergence belief systems contain diversity in terms of the subjects about which beliefs are held, and the range of beliefs that might be held about a particular entity. Redundancy exists as a consequence of the interconnections among beliefs. For example, derivative beliefs are not simply links in a chain originating in a primary belief. Rather, each belief in any such chain is likely also to be connected to other beliefs as parts of networks which are themselves connected to still other beliefs. Each of these beliefs may also be derivative so, although it may be possible to trace a chain from a derivative belief to a primary belief by repeatedly probing for the basis of each belief in the manner described by Green (1971), the chain that results is just one of many that could be identified. Some beliefs, in what is more like a dynamic network than a chain, are redundant in that the derived beliefs would not necessarily collapse if one or perhaps many related beliefs were no longer held. In addition, redundancy facilitates neighbour interactions by way of the connections between beliefs.

The dynamic behaviour of belief systems is not controlled by any central authority but happens unconsciously and can therefore be thought of as characterised by decentralised control. Much as Davis (2005) described the teacher in the complex system of a classroom as the consciousness of the collective, in relation to belief systems such attention directing activity places the conscious mind in the role of providing enabling constraints. In any given circumstance, certain beliefs may be called upon so that the activity of the belief system is not random but adaptive, that is, characterised by learning.

Influencing individual teachers

Consciousness can be seen as both shaping activity in the system and as an emergent characteristic of it. Its role can be seen as deciding which beliefs to privilege in a given context. It has been recognised that an individual's willingness to change is a prerequisite for change to occur (Carter & Norwood, 1997), and so there must be intentionality on the part of the individual who changes. Those interested in facilitating teacher change, therefore, need to consider ways in which individuals' belief systems might be influenced such that the intention to change (in desired directions) is likely to emerge. The chance of emergence occurring is in turn improved by increasing the system's diversity, redundancy, and connectivity (neighbour interactions). Things like: providing new ideas either from interactions with peers and/or experts or from materials (increased diversity); helping teachers to make links between beliefs that may be held in separate clusters by uncovering and challenging apparent contradictions among beliefs can all be seen as increasing the diversity, redundancy, and neighbour interactions within the individual's belief system.

Watchful anticipation (Seel, 2003) reinforces what we already know about the need for adequate time for teachers to adopt new perspectives and practices (Schleicher, 2016) and requires more than encouraging them to adopt new beliefs. Changing one's own belief system in significant ways even when one wants to can be difficult and require sustained effort. Carlisle and McMillan (2006) suggested that radical change in a belief system is likely when the system can be thought of as at "the edge of chaos" (p. 4). That is, when beliefs are highly connected, thereby increasing redundancy, the potential of the diversity present to result in emergent change is maximised. As noted earlier, among the most central of an individual's beliefs are those about him/herself and to which strong emotion is attached. Orchestrating circumstances that evoke a strong emotional response presents as a potentially powerful way to challenge these beliefs (e.g., Liljedahl, 2016). Briody, Pester, and Trotter (2012) discussed the use of stories to provoke organisational change. Because memories tend to be articulated as stories, and because of the emotional entailment of memory, stories that provoke or evoke strong emotional responses could be a powerful way to influence beliefs.

Influencing groups of teachers

Influencing the beliefs of a group of teachers, conceived of as a complex system that is emergent from the belief systems of the individual members, involves more than influencing as many individuals as possible. An individual teacher who has faced little that challenges her/his existing belief system is likely to have settled into a quite stable pattern of thinking and practice. Group contexts afford opportunities to increase the diversity in each of the individuals' belief systems as well as contributing to the diversity of beliefs that belong to the group. In a group, the belief systems of individuals can interact in such a way that a system of beliefs of the group that is not simply the sum of the parts of the individual's belief systems that are shared, can emerge (e.g., Beswick et al., 2010). From a complexity perspective, the facilitator of a professional learning (PL)session can be seen as establishing the conditions (enabling constraints) in which appropriate diversity and redundancy exist and neighbour interactions can occur. Whereas the challenge in influencing individual teachers might lie primarily in increasing the diversity of beliefs in the individual's system, a major challenge in a group context is to achieve sufficient redundancy in the shared collection of beliefs so that meaningful communication (neighbour interactions) is possible. That is, group members need to have sufficient understanding of one another's points of view in order to be persuaded by them or to challenge them. Diversity can also be enhanced by introducing new ideas to the group and facilitating interaction around these.

The more diverse the group members, the greater the potential for diversity among the ideas that drive interactions. Encouraging productive interaction among members of a group simultaneously fuels diversity, redundancy, and connectivity (neighbour interactions) within individual participants' belief systems. In effect, planning PL experiences mindful of the conditions for complexity in relation to the system comprising the participants in the activity, emphasises the dynamic, organic nature of the learning process and points to the value of considering both groups of teachers and the beliefs of individual teachers, as systems with properties and potentials that transcend the individual elements. Mowles (2014), in discussing the application of complexity science to evaluation scholarship, challenged evaluators to take seriously the fact that in intervening in a system to evaluate it, they necessarily change the system. Furthermore, since the emergent properties of complex systems are not predictable, traditional notions of objectives make little sense (Mowles, 2014). Similar arguments apply to teaching and to PL and suggest that the best we can do in both cases is to provide the conditions for change in particular directions.

A STUDY

Interview data from a larger study of the beliefs of individual teachers and the beliefs of a group of teachers is presented to illustrate some of the ideas discussed.

Interview

Audio-recorded, semi-structured interviews ranging of 30-60 minutes duration were conducted to explore teachers' beliefs relevant to their mathematics teaching. Questions related to: teachers' views about what it means to do mathematics and to think mathematically; their personal use of mathematics; the reasons for which some students experience difficulty with learning mathematics and how they respond to such difficulties; the extent and causes of students' disengagement with mathematics and their responses to it; their role in establishing appropriate classroom environments; and their participation in and commitment to PL. The teachers were also asked to respond to each of seven short paragraphs that aimed to unpack beliefs likely to be relevant. They were based upon those identified by Beswick (2007) and are shown in Figure 1.

1. Maths is fun: You have confidence in your ability to learn and to do mathematics and enjoy doing so. You believe that maths is interesting, you have confidence in your ability to learn and understand maths as required, and a delight in 'playing' with maths and mathematical ideas. Learning and teaching maths is an exciting adventure

2. Student achievement is predetermined: You see the students' maths achievement as a function of factors largely beyond the teacher's control. You believe that students are either good at maths or not, and that if they do poorly in one year then they are likely to do poorly in subsequent years. Some students are not capable of achieving conceptual understanding of the mathematics they are studying.

3. Teacher responsibility for learning: You believe in taking a proactive approach to teaching mathematics by creating the kind of classroom environments in which mathematical thinking is fostered and students learn to use mathematical conventions and to communicate mathematically. You believe that you can influence the achievement of all students and that you have a responsibility to intervene if learning is not occurring.

4. Non-intervention: You believe that there is little point in trying to get students to engage with the lesson if their interests are clearly elsewhere. There is in fact very little that a teacher can do to engage some students with mathematics. Provided students who are off-task are not disrupting the learning of others it makes sense to let them be so that you can focus on those who are keen to learn.

5. Sense-making: You see maths a powerful means of making sense of the world. Far from being simply a collection of skills and procedures you see mathematics as made up of ideas that are interconnected in complex and not necessarily hierarchical ways. You believe that recognising and exploring these interconnections is a vital part of learning mathematics.

6. Professional learning: You believe that you have a responsibility for your own professional learning in mathematics. To this end you believe that it is important to keep informed of the latest insights from research in mathematics education and to read widely about how students learn mathematics and how best to teach it.

7. Social norms: You believe in taking a proactive approach to establishing the kind of classroom environment in which students can feel safe to share their thinking and emerging understandings. To this end you believe in establishing and enforcing clear rules for interactions among students and between students and teacher. It is part of the teachers' responsibility to teach students appropriate ways of interacting.

Figure 1: Paragraphs to which teachers responded.

Participants

Eight teachers were interviewed. All but one had studied some kind of science at university, three indicated that they had a minor in mathematics and one had a mathematics major. Their teaching experience ranged from less than 1 year to 35 years. Leanne was selected as a particular focus in this paper because she appeared to hold beliefs that were indicative of those of the group generally, as well as others that contrasted with them. In addition, as the most experienced and mathematically qualified of the group, her case provides particular insights into the beliefs of such teachers who, in countries like Australia where there is considerable out-of-field mathematics teaching (McKenzie, Weldon, Rowley, Murphy & McMillan, 2014), might not be prioritised for PL and who are likely to have quite stable belief systems.

Data analysis

Similarly to Beswick (2015), interview transcripts for individual teachers were examined for statements of beliefs. These were extracted, sometimes with re-wording for clarity, and used as the basis for inferring more general beliefs that seemed to underpin the specific belief statements. Beliefs of the eight teachers as a group were derived from themes and subthemes that emerged from the collection of transcripts and the individual belief statements evident from them. They reflect beliefs that appeared to be held by the majority of the teachers. Importantly, they are not beliefs of the group in the sense of being emergent from the interaction of the belief systems of individual teachers, rather they are a collection of beliefs that many of the teachers held and that could form the basis on which a PL facilitator might begin the kinds of interactions that could lead to the emergence of group beliefs.

RESULTS

Findings from the eight interviews are summarised under three headings that represent broad themes from the data. Examples are provided of Leanne's responses.

Mathematics and mathematics teaching

Thinking mathematically was described in terms of precision, efficiency, and thoroughness. All but two teachers, including Leanne, contrasted mathematical thinking with creativity and described it as free from emotions. Leanne described doing mathematics from a student's point of view as being about learning techniques, answering questions, and passing exams, and for the teacher as being about trying to interest students in the hope that "one or two of them (might be) curious enough to say, 'Oh, what about this, does it follow on?'". She described thinking mathematically as logical and sequential and contrasted it with creativity and brainstorming.

All eight teachers expressed agreement with the Paragraphs 1 and 2 with Leanne explaining that she did not think it was always necessary for students, particularly those studying at more demanding levels, to understand how formulae that they used were derived. Overall, the teachers' saw mathematics as a sense-making and personally enjoyable activity, but many, including Leanne, were largely unable to separate their views of the discipline from their views of mathematics teaching.

Engaging students

The teachers talked about the responsibility that they had for ensuring that students learn and for establishing the kind of classroom environment in which this can happen primarily in their responses to Paragraphs 3, 4 and 7. Although many believed it was appropriate to ignore disengaged students, provided they were not being disruptive, this did not appear to be indicative of a laissez faire approach to teaching or of teachers abrogating their responsibilities, rather it seemed to be linked to teachers' beliefs about their efficacy with respect to engaging some students. Factors that they identified as contributing to a lack of belief in their efficacy included large class sizes, lack of parental and/or school and system level support, and the curriculum. There was some evidence that experience enhanced self-efficacy in this regard. Teachers reported and seemed to expect higher levels of engagement from classes that they perceived as mathematically capable but there was also evidence that this connection may have been mediated by different beliefs about the relative compliance of students in the different groups. Five of the teachers, but not Leanne, mentioned that the priorities in adolescents' lives revolved around their social lives and family circumstances.

Leanne's views on these matters were rather different from those of the other teachers, a fact that she explained by reference to the school context in which she worked. She qualified her agreement with Paragraph 3, explaining that she saw the students rather than herself as increasingly responsible for their learning as they progressed through the grade levels. In discussing Paragraph 7, she focussed on the need for rules to manage students' behaviour and explained that this was not an issue in her school, at least not in her classes, as it was an independent (i.e., fee-paying) school. Her response to Paragraph 4 was:

> I cannot teach in a room where somebody's not engaged. I've got to have them engaged. If I have somebody who's not engaged I will send them out of the classroom which we can do in our school and send them to the Dean's office rather than leaving them in the classroom because I believe having a student like that in the classroom is destructive to others.

Despite her adamant response to Paragraph 4, elsewhere Leanne's responses suggested that her high standards for student engagement applied less to students whom she perceived to be of lower ability. She explained that only 2-5% of her Grade 10 class were engaged with mathematics and indicated that maintaining the focus of the lesson was less important for these students. She said:

> If we've got too many away they have talked me into playing monopoly because the bottom Grade 8 and Grade 9 teacher has monopoly games there so they can learn about money and counting and so on.
>
> The disinterest is expected at the bottom [ability level] Grade 10 level and therefore I try to design the curriculum to get something that they can see the relevance (of) … I don't see the disinterest in Grade 8 or 9 levels because I teach the top [ability level] class.
>
> I find that with the top groups they only have to be told once. They're very good.

A further comment suggested that the differences in her teaching depending on her perception of the ability of the class may have been related to her beliefs about the teaching styles that these different groups respond to. In the context of talking about the reasons for students not engaging with mathematics, she described another teacher, "whose teaching method was harsh and disciplinarian type and the students' response was to turn off".

Leanne and two other teachers also referred to students' concern with the relevance of the mathematics they learned. These comments were all made in the context of talking about students whom they described as being in their lower ability classes or who found mathematics difficult. Leanne and another teacher also linked disengagement with students struggling with the work, explaining that some students will disengage rather than admit that they are struggling. Leanne was also one of three teachers who mentioned their use of extrinsic rewards to keep students on task.

Student ability

The data suggests that many teachers considered ability to be fixed to some extent, although only four, including Leanne, explicitly mentioned 'ability'. It seemed that a fixed view of ability influenced the beliefs of some of them about the feasibility of teaching for conceptual understanding. All of the teachers interviewed expressed the belief that they could influence their students' achievements, but two of them, including Leanne, also cited experiences of students failing to learn in spite of their best efforts. Leanne was unequivocal in agreeing with the statement in Paragraph 2 that "Some students are not capable of achieving conceptual understanding". Her beliefs in fixed ability were conveyed in statements such as:

> I have seen students who will make the same mistake over and over and over no matter how simply you explain it. They do not seem able to grasp the concept. If they can grasp the concept they will forget it again tomorrow and the forgetting part I think is that ability.

> There are people who cannot do mathematics. I can't do art … I think mathematical ability exists in the same way artistic ability exists … so there are children who will all the time have trouble coping with numeracy.

In the sections that follow aspects of first Leanne's belief system and then of the group of teachers are described. In both cases the beliefs shown are illustrative.

Aspects of Leanne's belief system

Figure 2 shows a selection of belief statements that are derived from Leanne's interview and organised according to three broad themes. Overlapping regions contain belief statements that seem to belong in both/all of the overlapping categories. Although it could be argued that particular beliefs might be better placed in different regions, it is clear that there are connections among the ways in which Leanne thought about various aspects of her mathematics teaching and her students. As a result, there is likely to be considerable redundancy around, for example, her beliefs about students' ability, meaning that if any particular of these beliefs was challenged by evidence either from outside of Leanne or from her own experience, its many connections within the system would make it resistant to change. For example, if Leanne was presented with evidence that memory is not necessarily related to mathematical ability she would likely have sufficient examples of students from her experience whom she had categorised as low ability and who did have trouble remembering procedures that she would be able to accommodate the new information as a special case. Even if she was somehow persuaded that memory and mathematical ability are largely and generally separate, the impact on her teaching might still be minimal given other central beliefs. For example, the belief that "top level students just need to be able to use the rules" points to a strong role for memory in success for these students. The association between high ability and memory might thus be modified but not entirely broken.

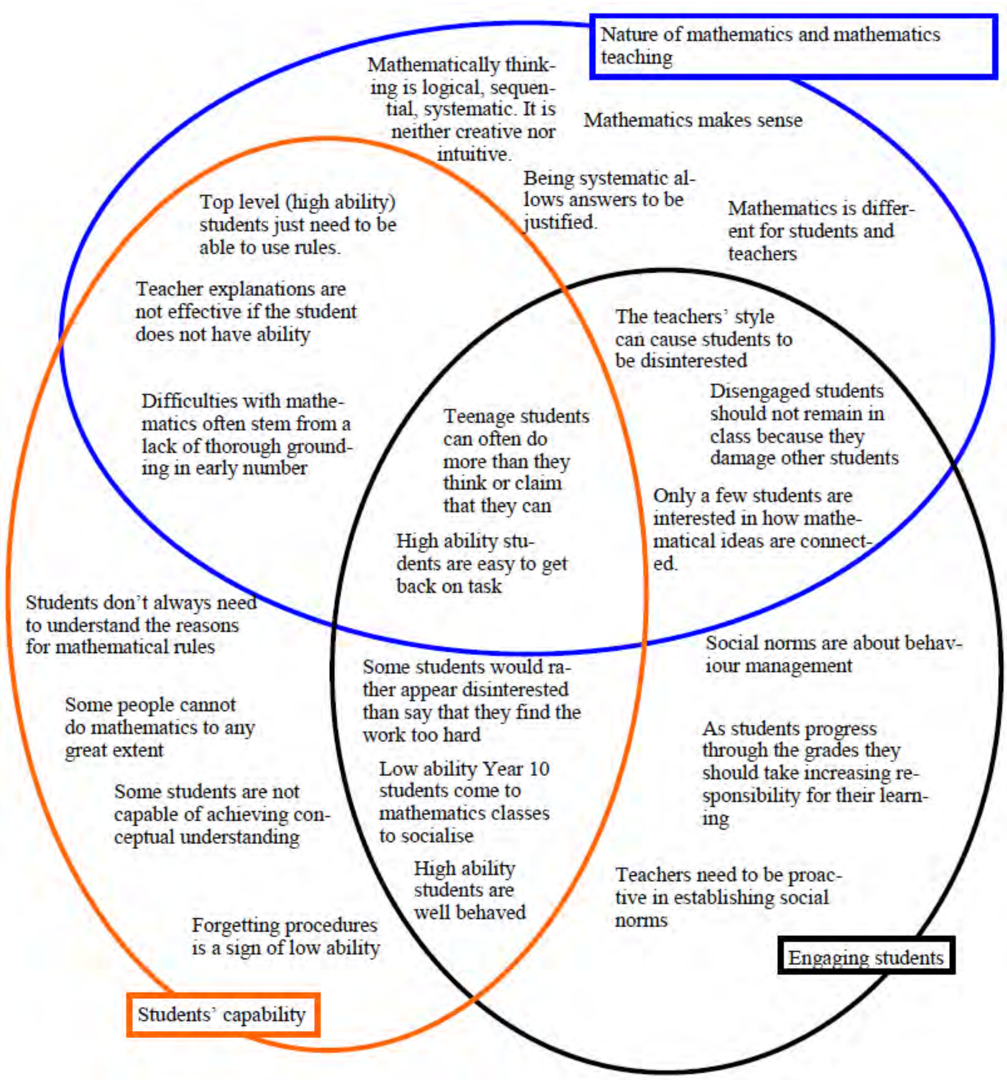

Figure 2: Aspects of Leanne's belief system.

Among the most central of any individual's beliefs are those about themselves. Their centrality stems in part from their primacy (Pajares, 1992). Although Leanne did not explicitly characterise herself mathematically capable, the context in which several of the beliefs shown in Figure 2 arose suggests that they are likely to be linked to Leanne's beliefs about herself. These include beliefs as "Teacher explanations are not effective if the student does not have ability" and "The teacher's style can cause students to be disinterested." The first of these was in the context of having done her best. The second was connected with an account of a colleague for whom this was true, perhaps implying that it did not apply to her. In addition, her claim that she "cannot do art" implies a belief that she can do mathematics. Together these beliefs suggest that Leanne saw herself as someone who is quite good at mathematics and perhaps at being logical and systematic generally, and as an effective mathematics teacher. Believing that one is good at mathematics as a central part of one's belief system necessitates believing that there exist others who are not good at mathematics. Similarly, maintaining belief in oneself as a good teacher in the face of students who do not appear to learn is likely to lead to seeking explanations for student difficulties that are beyond the reach of teaching—in, for example, the innate nature of the students' ability, prior teaching, lack of self-belief, or prioritising socialising, all of which were evident among Leanne's beliefs.

'Shifting' Leanne's pedagogy would be challenging: Her belief system seems highly integrated, coherent, and stable. In complexity terms there is relatively little diversity among her beliefs to balance the high level of redundancy. Any change would require a conscious desire on the part of Leanne to change. Motivation to do so would appear to require her beliefs interacting with others not currently part of her system and in a way that would undermine a large number of highly central beliefs while allowing her to maintain her sense of self.

Aspects of the teachers' belief systems

Figure 3 represents a selection of beliefs shared by at least three of the eight teachers grouped according to the same themes as in Figure 2.

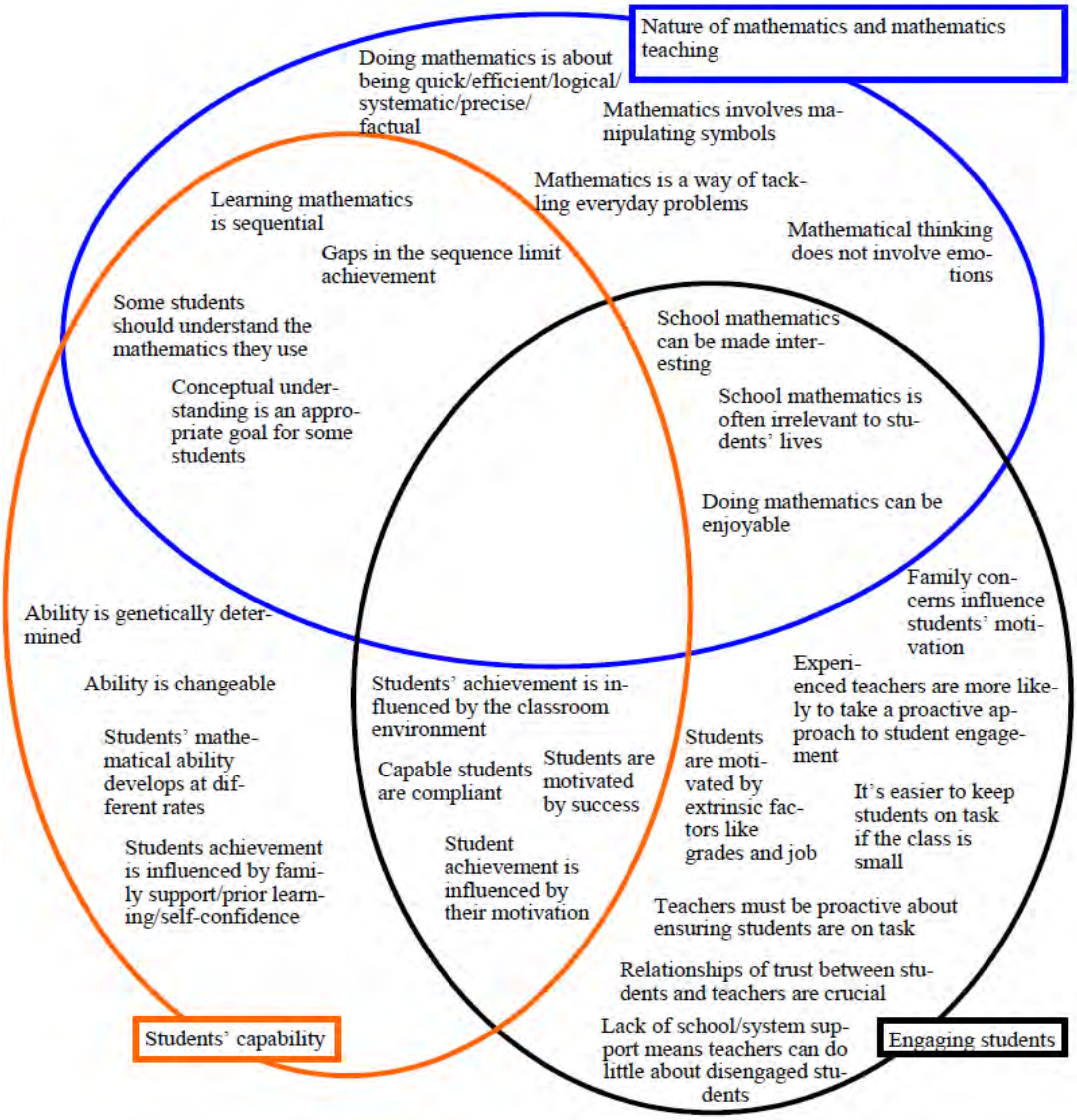

Figure 3: Aspects of the group's beliefs.

As was the case with Leanne, the overlap among the sets of beliefs points to a redundancy among them. Similar arguments to those made in relation to Leanne could be made about many of these teachers' beliefs about themselves and about students. There are, however, at least two places that suggest that, as a group, the beliefs of these teachers are (unsurprisingly) less coherent than those of Leanne. First, there is a diversity of beliefs about the extent to which mathematical ability is innate and fixed. This presents the possibility that surfacing these beliefs and facilitating their 'bumping together' by way of robust discussion (neighbour interactions) could contribute to change for some of the teachers and/or the emergence of shared beliefs. Second, the beliefs that "mathematics can be made interesting" and "doing mathematics can be enjoyable" suggest that experiences that tap into these beliefs and surface the diversity among them around the extent to which these things are possible and, hence, the extent to which mathematics is inherently enjoyable and interesting, could be generative.

CONCLUSION

Viewing the belief systems of both individual teachers and groups of teachers as complex systems normalises the unpredictable impacts of efforts to influence them while pointing to ways in which professional learning (PL) activities might be more effective. Just as we urge teachers to take account of the current thinking of their students, it behoves PL facilitators to spend time exploring the beliefs that teachers bring to the experience and to devising ways that individual beliefs can be surfaced and shared to fuel activity and change in both the belief systems of individual participants and the emergence of group beliefs. A complex system perspective emphasises the value of working with diverse groups of teachers and of maximising robust interactions among differing beliefs.

The role of emotions and memory in catalysing change was not strongly evident in the data presented here. It would be possible, however, to analyse recordings of interviews for emotional charge attached to various belief statements. This might provide useful insights into the relative centrality of the beliefs expressed. There is certainly more to be done on the use of emotionally charged experiences and stories in catalysing beliefs change (e.g., Liljedahl, 2016). Part of uncovering the existing beliefs of teachers might be encouraging them to recount experiences of learning mathematics including recalling the associated emotions and helping them to make connections between these stories and their beliefs about mathematics teaching.

The study reported here adds to existing evidence that many mathematics teachers hold fixed beliefs about the capacity of certain of their students to learn mathematics and about the kinds of learning experiences that are appropriate for various students. We know that beliefs that certain students or classes are of low mathematical ability is likely to condemn these students to low level academic expectations (Gervasoni & Lindenskov, 2011) and impoverished curricula (Beswick 2007/2008). That ability grouping persists in many school systems in spite of evidence of its counterproductive effects points to the importance of influencing teachers' beliefs about students' capacity to learn. This study provides further evidence of the entailment of beliefs about the capacities of students to learn with teachers' beliefs about themselves as 'good at' mathematics and their ability to teach low attaining students effectively. Change in this area is an example of a hitherto intractable problem in mathematics education that demands a system view of change.

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FROM ANXIETY TO ENGAGEMENT: HISTORY AND FUTURE OF RESEARCH ON MATHEMATICS-RELATED AFFECT

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Anxiety is an important concept in mathematics education. It has roots in psychoanalytic theory, behaviourism, and social psychology. In mathematics education, other early areas of research were emotions in problem solving, beliefs, and attitudes. Research on mathematics-related affect has struggled to find a shared theoretical framework. Another problem with this research is too strong focus on individual trait type affect, and more research on group level dynamic processes is needed. Some examples of research on collaborative engagement are presented, including first results of a mobile gaze tracking study.

INTRODUCTION

In this plenary, I will trace some historical threads of research on mathematics-related affect. Research on anxiety has influenced largely later research, anxiety being both a dimension of mathematics attitudes and an emotion related to mathematical problem solving. Therefore, I will dig a little deeper for the origins of anxiety as a research topic in psychology and education. From there, the story will continue at a faster pace through the expansion of research from anxiety to a variety of attitudes and emotions. Finally, I will explore some of the current research on engagement with a specific focus on collaborative engagement.

ANXIETY IN ACTION

I will illustrate what mathematics anxiety is and how it influences learning through a case study from my doctoral thesis, which was published in *For the Learning of Mathematics* (Hannula, 2003). All excerpts below come from that article (pp. 33–35).

Helena was a grade nine student in a class where I was doing a three-year ethnographic study. Helena was a diligent student, but her achievement in mathematics was not good. In an interview, Helena explained her fear for mathematics:

> Helena: Mathematics makes me […] anxious […] mood goes really down […] and then just wait for the next exam horrified.

Specifically, she felt that she could not do problem solving:

> Helena: I have said to my mother, that I feel that I have a compartment in my brain that locks everything like that. [...]For me that is exactly what makes me feel anxious and […] somewhat unpleasant feeling that, sort of "Am I stupid or what – or feebleminded?", 'coz everyone else can solve this. So why can't I?

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 19-34). Umeå, Sweden: PME.

During one of the mathematics lessons that I observed, I had an opportunity to follow closely an illuminating incident with Helena and her teacher. Helena had not been able to do her homework. At the beginning of the lesson Helena and several other students asked for the teacher's assistance. When the teacher came to help Helena, he spent 15 minutes working with her while several other students waited for teacher assistance.

When the teacher began to explain, Helena seemed to concentrate hard. She reacted,

Helena: Oh, […]. I thought that…

The teacher interrupted her and continued explaining. A little later Helena misunderstood an instruction, and the teacher reacted rapidly and bluntly,

Teacher: No, no, no. Not 'k', but the value of k.

It looked as if Helena was shrinking in size as she bent downwards. Helena continued writing as dictated, erasing occasionally. When the teacher asked if Helena understood, she gave no answer. The teacher continued explaining and asking questions and gradually Helena's thinking seemed to deteriorate and eventually she was suggesting negative four times four to be four or plus two. In Helena's voice I sensed her distress.

When the teacher and Helena had finished the task, Helena began to solve another similar task. She chatted with her friend Jaana and they smiled. A while later they were even laughing. But she did not get any further with the task than she had at home.

After the 90-minute lesson, I had a short interview with Helena. First she told that she didn't "feel any special" at the time her teacher was helping her. When I said that to me it seemed much like one of the distressing events she told about in an earlier interview, she suddenly burst into tears:

Helena: I feel that it's no use for me to come to lessons, because I do not understand anything, and I never …

One striking thing in this episode was the level of emotional pressure Helena kept during the whole lesson—for over an hour. She was chatting and smiling with her friend to cope with her anxiety. Yet underneath the surface she was feeling so bad, that being reminded of the incident caused her to burst into tears. Also noteworthy is her reluctance to reveal her bad feelings, when asked about her situation.

This qualitative case study hopefully serves to illustrate mathematics anxiety as a phenomenon that is significant for the well-being and learning, for anyone experiencing it. Mathematics anxiety is a much researched area in mathematics education. Google Scholar (https://scholar.google.fi) identifies 4,290 results since 2014—about thousand new studies each year. Let us next turn to the history of this concept.

HISTORY OF ANXIETY RESEARCH

Many of the things that we still consider relevant with respect to mathematics anxiety have their roots surprisingly far in the history of psychological research. The three main areas of anxiety research were in psychoanalysis, behaviourism, and social psychology.

Beginning by the end of the 19th century and gaining momentum in the early 20th century, anxiety was an important concept for psychoanalytic research (e.g., Freud, Strachey, Freud, & Tyson, 1981). Freud identified realistic anxiety as fear towards an expected real danger and neurotic anxiety as a similar emotion without an obvious reason. Psychoanalytic research was based mostly on clinical practice and it was interested in neurotic anxiety as a form of psychopathology: a conflict between id and ego; a fear of unconscious material entering consciousness.

Psychoanalytic perspectives are used to some extent also in more recent research on mathematics anxiety. For example, Evans, Morgan, and Tsatsaroni (2006) paid attention to 'Freudian slips' and denial as indicators for "psychic defences against strong emotion" and identification with or resistance to a person providing psychoanalytical insights.

I argue that the psychoanalytical ideas shed some light also on the case of Helena. The core of her anxiety could be seen in her fear of being stupid—or even feeble-minded. Becoming conscious of this threatened her ego. Or, if one is not willing to use the psychoanalytic terminology, there was a threat to her self-esteem as student (also McLeod, 1992 suggested this as a cause for anxiety). When asked about her feelings, Helena's initial response was a denial.

Based on psychoanalytical research, psychodynamic therapy was developed as a treatment for many disorders, including anxiety disorder. Leichsenring, Hiller, Weissberg, and Leibing (2006) describe that, in psychodynamic therapy, the therapist helps the patient to gain "insight about repetitive conflicts sustaining his or her problems" (p. 238) as well as to "strengthen abilities that are temporarily inaccessible because of acute stress (e.g., traumatic events) or that have not been sufficiently developed" (p. 238). There is evidence for the efficacy of psychodynamic therapy in anxiety disorders (Leichsenring et al., 2015).

Although I could not find examples of psychodynamic therapy being used to treat mathematics anxiety, we can find traces of it, especially in research on mathematics anxiety of adult learners in teacher education. For example, through studies on student affect in Finnish elementary teacher education, Kaasila has developed narrative rehabilitation and bibliotherapy as methods to help pre-service teachers build a more positive mathematical identity as a future teacher (e.g., Lutovac & Kaasila, 2011). When these students write and share their mathematical autobiographies or read accounts they can identify with, they are involved in similar processes of self-reflection and therapeutic support as in psychodynamic therapy.

Along with psychoanalysis, the gradually strengthening experimental psychology, following the behaviourist paradigm, also explored the role of anxiety as a learned response (e.g., Mowrer, 1939; Estes & Skinner, 1941). Anxiety disorder was one of the maladaptive learned response patterns, and behavioural therapy would focus on removing the response through conditioning (Eysenck, 1959). In stark contrast to the psychoanalytic school, the behaviourists did not assume any unconscious causes for the disorders, and the focus in behavioural therapies was on symptoms: "*Get rid of the symptom and you have eliminated the neurosis*" (Eysenck, 1959, p. 6). Systematic desensitization, a behavioural therapy based on exposure and relaxation, has been proven effective for anxiety disorders (e.g., Berman, Miller, & Massman, 1985), including mathematics anxiety (Hembree, 1990).

The third and the most important foundation for mathematics anxiety research was social psychology research on attitudes (Thurstone, 1931). In this research approach, the interest was not on the emotional disorders, but on the role that attitudes had on behaviour in the larger population. Attitude research was important for developing the methodology for survey instruments. Based on a review of the early studies on attitudes towards mathematics, Biggs (1959) summarized the most common reasons for negative attitudes to be "lack of a proper understanding of the subject, which is in many cases due to mechanical teaching; and upon plain boredom, a product of uninteresting teaching" (p. 20).

Test anxiety is the closest predecessor to mathematics anxiety. Sarason and Mandler (1952) initiated this line of research coinciding with a transition from behaviourism to cognitive science. Their conclusions include an observation that test scores do not adequately describe the abilities of anxious individuals as their anxiety interfered with task-relevant efforts. Although Helena in our case study was not in a test situation, we can see her cognitive performance falling below her normal level.

Dreger and Aiken (1957) were the first to focus their research on mathematics as a specific area of anxiety, calling it number anxiety. It seems to have been a phenomenon well known by practitioners even earlier, as indicated by insightful articles by experienced practitioners: *Arithmetics without fear* (Rogers, 1940) and *Why Failures in Mathematics? Mathemaphobia: Causes and treatments* (Gough, 1954). Two years later Biggs (1959) wrote that mathematics anxiety was recognised by the general public as probably the greatest problem in the teaching of mathematics. Yet, research on mathematics anxiety gained momentum only after Richardson and Suinn (1972) published the Mathematics Anxiety Rating Scale.

An important theoretical distinction was later made between two types of anxiety: "anxiety as a transitory state that fluctuates over time and as a personality trait that remains relatively stable over time" (Spielberger, 1966, p. 15). In the case of Helena, she expressed her *trait* anxiety in the interview, but when the teacher was helping her, she was experiencing *state* anxiety.

Reviews and meta-analyses of research on mathematics anxiety (e.g., Dowker, Sarkar, & Looi, 2016; Hembree, 1990) have shown that mathematics anxiety has a negative correlation with mathematics attainment and positive attitudes toward mathematics; is correlated with, but distinct from, general anxiety and text anxiety; and is more common among females than males.

FROM ANXIETY TO AFFECTIVE STRUCTURES

I am going to fast-forward the next decades of research so as not to repeat what I have written earlier (e.g., Hannula, 2012; Hannula 2015). McLeod (1992) summarized most of the research done by the end of 1980's, which consisted of three main areas: emotions, attitudes, and beliefs.

One area of McLeod's (1992) review was research on emotions during mathematical problem solving. Already Polya (1957) addressed this topic, and the role of affective variables was extensively elaborated in the 1980's, when Mason, Burton, and Stacey (1982), Schoenfeld (1985), McLeod (1988), Goldin (1988), and Cobb, Yackel, and Wood (1989) all let affect play a significant role in their analyses of mathematical problem solving. One conclusion of this research is that both negative and positive emotions can be essential for the problem solver's self-regulation (Hannula, 2015).

The main bulk of research in mathematics-related affect in McLeod's (1992) review consisted of surveys about *attitudes* (e.g., anxiety and liking mathematics), and *beliefs* (e.g,. about self, mathematics, learning, and teaching). These studies identified, for example, a gender difference favouring males, that mathematics achievement correlates positively with positive attitudes and beliefs, and an overall tendency for students' relations with mathematics to become more negative over the school years.

The relation between affect and achievement

While the positive correlation between mathematics-related affect and mathematics performance was established early on, the direction of causality has been more problematic to determine. My colleagues and me (Hannula, Bofah, Tuohilampi, & Metsämuuronen, 2014) reviewed longitudinal studies on the relationship between affect and achievement in mathematics from Australia (Green, Liem, Martin, Colmar, Marsh & McInerney, 2012; Seaton, Parker, Marsh, Craven & Yeung, 2013), Finland (Hannula, Maijala, & Pehkonen, 2004), Italy (Capara, Vecchione, Alessandri, Gerbino, & Barbaranelli, 2011), Japan (Minato & Kamada, 1996), and the USA (Ma & Xu, 2004). These longitudinal studies provided

> (…) strong evidence for a reciprocal relationship between academic self-efficacy and achievement. There is mixed evidence for the dominant direction of this relationship and for its development. With respect to the relationship between mathematics-related emotions and achievement the evidence is even less clear, but it suggests a reciprocal linkage, with the dominant direction possibly from emotions to achievement. (Hannula et al., 2014, p. 251)

In our analysis of nationally representative longitudinal data (grades 3, 6, and 9) from Finland, we were able to further study the relationships between achievement in mathematics and two affective measures: *enjoyment of mathematics* and *self-efficacy in mathematics* (Hannula et al., 2014). The results indicate that mathematics achievement and self-efficacy have a reciprocal relation, where the dominant effect is from achievement to self-efficacy; the effect of self-efficacy on achievement was growing larger for the older students; and there was also a weaker unidirectional effect from achievement to enjoyment.

However, the same group of students was studied also at grade 12, and these results indicate that student affect was as equally important a predictor as their achievement, for how much they chose to study mathematics after grade 9, which again was highly influential to their mathematics achievement at the end of secondary school (Metsämuuronen, 2017). Taken together, these results suggest that student affect has only a limited effect on student learning of mathematics as long as studying it is compulsory, yet student affect becomes much a more important factor as soon as the student can opt out of mathematics.

Structuring the affective area

Research in mathematics education has introduced many affective concepts. For a review chapter on research on affect in the CERME conferences we analysed the terminology for affect appearing in 134 published conference papers, identifying 51 different terms in the titles alone (Hannula, Pantziara, & Di Martino, 2018). Discussions about the terminology and quest for a shared theoretical framework for mathematics-related affect have been ongoing (e.g., Goldin et al., 2016; Hannula, 2012; Hannula et al., 2018; McLeod, 1992; Furinghetti & Pehkonen, 2002; Zan, Brown, Evans, & Hannula, 2006).

In this complexity of concepts, identifying the relationships between different affecttive variables would be a way to find structure in the chaos. Instruments tapping multiple affect constructs that we developed and used with Finnish population were promising as similar structures were identified for grades 4 and 8 (Hannula & Laakso, 2011), 11 (Roesken, Hannula, & Pehkonen, 2011), and elementary education students (Hannula, Kaasila, Laine, & Pehkonen, 2006). These studies indicated a trend of increasing reliabilities for affect variables and increasing correlations between affect variables as students grew older, which suggests that the affective structure is in a process of formation during adolescence.

However, this line of research proved to be more difficult than anticipated when we started making cross-cultural comparative studies. When instruments validated in Finland, or parts of them, were implemented in Estonia (Kaldo & Hannula, 2012), Chile (Tuohilampi, Hannula, Varas, Giaconi, Laine, Näveri, & i Nevado, 2015), and Ghana (Bofah & Hannula, 2015), at least some of the scales failed to work in the new context. This suggests that even the components of mathematics-related affect are not universal.

A metatheory for affect

Although the structure of affect seems not to be universal, it is possible to structure the research domain on a more abstract level. I have suggested (Hannula, 2012) that the plethora of theoretical frameworks for affect can be organized along three dimensions: 1) The type of affect being either cognitive (beliefs), motivational, or emotional; 2)

The focus of analysis either being in the rapidly changing affective states or the relatively stable affective traits; and 3) The theoretical frame focusing either on physiological (embodied), psychological, or social level of affect (Figure 1).

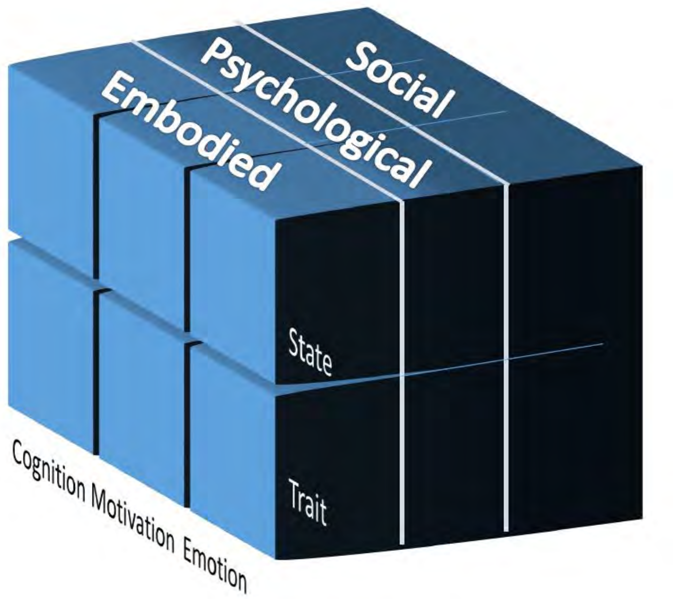

Figure 1: The cube model for the metatheory of affect (Hannula, 2012).

For example, mathematics anxiety is an emotion that has been examined both as a trait and as a state (Hembree, 1990). While mathematics anxiety is typically conceptualized through psychological theories, there is also active neurophysiological research on mathematics anxiety. For example, Sarkar, Dowker, & Kadosh (2014) reported a study, where brain stimulation improved reaction times on simple arithmetic decisions and decreased cortisol concentrations in high mathematics anxiety individuals, while for low mathematics anxiety individuals the stimulation impaired reaction times and prevented a decrease in cortisol concentration.

When reflecting on the field of affect in mathematics education research (Hannula, 2012) it became apparent that there were many more studies that focus on *traits* than those focusing on *states*, and that a psychological approach was dominating over social or embodied perspectives. In particular, few studies focused on the dynamics of emotional or motivational states in a classroom.

The research area on affect as a dynamic social state is a fruitful and interesting direction for many reasons. First, this direction is poorly explored. Secondly, the goals and recommended practices of schools and workplaces of today and tomorrow focus on collaboration. Thirdly, understanding the affective-social processes in the class-room will help us create classrooms that are able to emotionally engage our students and build their motivation. Fourthly, it's about time. Already sixty years ago Johnson (1957) complained in a review about the lack of research in these areas:

> However, there was no research found on teaching children how to work together, how to plan, or how to evaluate what they have learned. These are goals modern education has accepted as being as important as facts, skills, and problem solving. (p. 409)

ENGAGEMENT

We now move to explore engagement as an example of a dynamic, positive emotion on the individual level, then continue to see how it might be studied on the group level.

Problem solving and flow

One interesting approach to the dynamics of positive emotions is *flow*, an intensive level of engagement. When students find an optimal balance between skill and challenge, they can experience flow (Nakamura & Csikszentmihalyi, 2014). If the challenge is too high they may experience *anxiety*, and when the challenge is too low they may experience *boredom*. We know that both anxiety (Dowker et al., 2016) and boredom (Tze, Daniels, & Klassen, 2016) are common in mathematics lessons.

Liljedahl (2016b) points out that while students are learning, their skills improve and, therefore, also the challenge must increase to make flow possible, but the challenge may not increase too rapidly, or the student might get anxious (Figure 2). This figure is a simplification, and there are a number of other possible emotional states, such as control and relaxation (between flow and boredom) and arousal (between flow and anxiety) (Nakamura & Csikszentmihalyi, 2014).

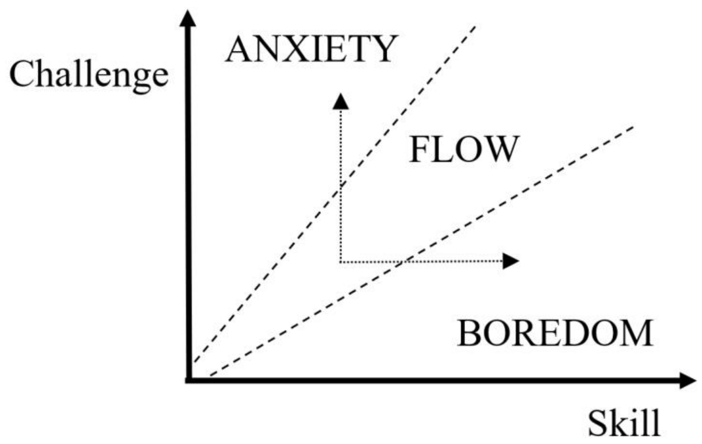

Figure 2: Flow as optimal challenge and the paths from flow to anxiety and boredom.

Using a problem solving approach, it is possible to create a mathematics class that is full of engagement:

> Children frequently jumped up and down, hugged each other, and rushed off to tell the teacher when they solved a particularly challenging problem. Significantly, the positive emotional acts occurred when the children completed personally challenging tasks or constructed mathematical relationships. (Cobb et al., 1989, p. 61)

In a three-year research project we explored the overall effect of spending just one mathematics lesson per month on open problems with multiple answers on students' affect and achievement (Laine, Näveri, Ahtee, Pehkonen, & Hannula, 2018). Using a survey approach, we could see some positive effect of the intervention: female students' self-confidence and enjoyment of mathematics did stay more positive in comparison to the control group (Tuohilampi, Näveri, & Laine, 2015). It can be argued that an intervention using only one lesson per month was probably not powerful enough to have a stronger impact. However, both, student drawings of their mathematics lessons (Laine, Ahtee, Näveri, Pehkonen, Portaankorva-Koivisto, & Tuohilampi, 2015), and our closer analysis of teacher actions during some intervention lessons (e.g., Laine et al., 2018) indicated a lot of variation across the actual practices of different intervention classrooms. Even if there was increased engagement in all intervention classes, the nature of social engagement seemed to be different.

Although it seems that a high level of engagement is commonly achieved using problem solving, it is not automatically successful. When Liljedahl (2016a) describes how ordinary mathematics classes are transformed into "thinking classrooms", he also shares his less successful first attempts. To create a thinking classroom, he argues that good problems are necessary, but not enough. The social structures in the class need to be also reorganized and to make this happen students are organized in visibly randomized groups and they work on vertical non-permanent surfaces. When these and some other changes are implemented, students are much more engaged than before.

Yet, even if the implementation of engaging classrooms is successful, there are details to explore. When students are engaged, what are the different ways in which they engage? Are some of these better for learning than others? How do students influence each other's engagement? Moreover, how do we conceptualize and investigate the emotional-motivational collaborative processes on the group level? To answer these and other questions, we need to move on to study what happens in an engaged mathematics classroom. Then we can continue developing engaged mathematics classrooms.

MathTrack – tracking the collaborative engagement

In our current research project MathTrack (Academy of Finland grant 297856), we use mobile gaze tracking to analyse the visual attention of teacher and students during collaborative problem solving. For the purposes of our study, we are interested in students discussing, justifying, and negotiating about different solutions, that is, how they engage in the collaborative problem solving.

To track student visual attention during the collaborative problem solving, we have used mobile gaze tracking devices and algorithms developed at the Finnish Institute of Occupational Health and released as open source (Toivanen, Lukander, & Puolamäki, 2017). In addition, we have recorded classroom actions using regular video cameras, and student work using either smart pens or screen capture videos. In our research setting, four volunteering students and the teacher wear a mobile gaze tracking device. When all devices were working properly, we generated simultaneously five gaze tracking videos, three external videos, and four videos recording student work. Altogether, that is 12 synchronized videos for each class. In addition, we have video-stimulated recall interviews from the teacher and the four students wearing the gaze-trackers and some questionnaire data from all students and the teacher.

The coding of gaze targets manually is painfully laborious, but once it is done, we can see how the student navigates the different visual channels in the learning environment. For example, we can trace the student gaze from being stuck to having a sudden insight to analyse which of the several drawings in the group have triggered the new productive ideas. We can also examine how the listening student combines peer's verbal explanation with gestures and visual material.

Visual attention is also a wonderful method for examining student collaborative engagement. Once we have recorded the location of each student's gaze, we can also identify when the students were looking at the same target, giving us a good measure for collaboration. In Figure 3 we can see a problem solving sequence of three students who alternate between collaborative and individual work modes. The details of the method and context will be presented in future, (Garcia Moreno-Esteva, Salminen-Saari, & Hannula, Manuscript).

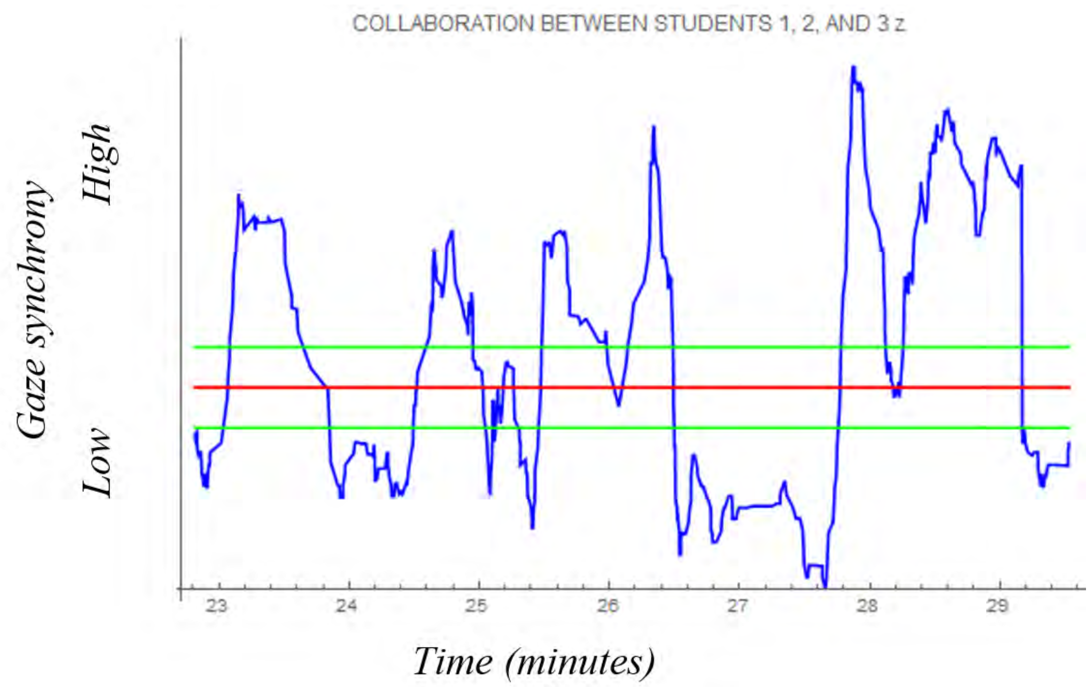

Figure 3: Graph for gaze synchrony for three students during problem solving. The horizontal axis is time (minutes), the vertical axis is gaze synchrony. The horizontal line in the middle is the baseline synchrony, the two lines above and below it mark the standard deviation. See Garcia Moreno-Esteva et al. (Manuscript) for details.

CONCLUDING REFLECTIONS

The modern era has been characterized by science and technology, mathematics being an important part of advancing human knowledge. Before the time of computers, mathematics was done by people and that often included complex computations. For example, Langley Memorial Aeronautical Laboratory had thousands of mathematicians—mostly female and including African-American—doing the necessary computations for aerospace research (McLennan & Gainer, 2012). The recent movie *Hidden Figures* (Gigliotti, Cherning, Topping, Williams, Melfi, & Melfi, 2016) tells of this era "when computers wore skirts".

Even when the space program had electronic computers, the astronauts still relied more on the human computers. Katherine Johnson, who was responsible for computing the trajectories told about this in an interview:

> But when they went to computers, they called over and said, 'tell her to check and see if the computer trajectory they had calculated was correct.' So I checked it and it was correct. (Hodges, 2008)

At that time, computing was literally a matter of life and death. Not only for the early space exploration, but errors in mathematical computations were potentially lethal also when building bridges and buildings. Error management was a central task, for example, in the Manhattan project developing the first nuclear weapon (Rall, 2006). At that time, accuracy and reliability were key requirements for many mathematical jobs. For example, Johnson (1957) wrote that "Practice and drill continue to serve the functions of increasing the retention, efficiency, and accuracy of skill learning in mathematics and science" (p. 407).

It has been suggested that although anxiety impairs performance on difficult tasks, it might even enhance performance on easy tasks (Boekaerts & Pekrun, 2015). Research indicates also that people under stress are better able to inhibit their instinctive responses and to select a more appropriate behaviour (Shields, Sazma, & Yonelinas, 2016). Therefore, it may have been to some extent a reasonable teaching strategy to emphasize accuracy in mathematics and anxiety of some individuals has been an acceptable collateral damage.

However, the needs of the 21st century are different. The focus also for mathematics education should be on critical thinking, communication, collaboration, and creativity. Such skills cannot be forced on anyone, they can only be learned through engagement with inspiring tasks and discussions.

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THE VERY MULTI-FACETED NATURE OF MATHEMATICS EDUCATION RESEARCH

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This paper is an analytic position paper. Its purpose is to identify and discuss a phenomenon that has evolved in mathematics education research journal publications during the last three decades, namely the emergence of an 'ideal-typical' research paper that represents a far too narrow and rigid understanding of mathematics education research by not reflecting the multi-faceted reality of the field as it has developed since the 1960s and as represented in other kinds of publications, primarily books. The paper offers an outline and an explanation of this development, and suggests that it is detrimental to a field which is far from having found a universal, agreed-upon theoretical underpinning. It argues that we have to take strong measures to prevent our field from prematurely congealing into insufficient research paradigms.

INTRODUCTION

It might seem tempting to claim that mathematics education as a research domain celebrates its fiftieth anniversary here in 2018. Along with a host of other significant undertaking and events around 1970, the world's first journal devoted to the scholarly and scientific study of mathematics education, *Educational Studies in Mathematics*, was established in 1968. The first *International Congress on Mathematical Instruction (ICME-1)* was held in 1969, and *Journal for Research in Mathematics Education* was established in 1970, as was the *International Journal of Mathematical Education in Science and Technology*. The very body under whose auspices we are gathering here in Umeå, the *International Group for the Psychology of Mathematics Education, PME*, was founded in 1976, in continuation of seminal activities at ICMEs 1 and 2 in 1969 and 1972, and was the first study group to be affiliated to *ICMI, The International Commission on Mathematical Instruction*. Depending, of course, on what notion of research one adopts, in general or with particular regard to mathematics education, there was also mathematics education research to be found before 1968. Conversely, one might well ask which of the papers published in ESM in 1968 and JRME in 1970 would pass as research papers according to the predominant perceptions of research today.

During the past fifty years, what has been published under the label of mathematics education research constitutes a very broad, diverse and extensive body of contributions to the field. And not only that, these contributions display an immense variation in *nature*. However, during the last around three decades I have detected a gradually reduced spectrum of variation in the nature of "mainstream" published research,

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especially in our journals. Although I do not want to take this too far, it sometimes seems as if we are moving in the direction of that kind of rather focused, perhaps even monolithic, perception of research that is characteristic of an old and established discipline having undergone centuries of development which has clarified its foundations and purified its methodologies and has resulted in a rather uniform standard for research publications and for the kind of research that lies behind them. However, is mathematics education today really such a type of discipline?

In this paper, I shall offer an account of this development, attempt to explain it, and finally offer a critical analysis of the consequences. By saying this I have already indicated that I have strong reservations towards aspects of this development.

ASPECTS OF THE DEVELOPMENT OF MATHEMATICS EDUCATION RESEARCH

If we look at papers published in ESM and JRME around 1970—the two oldest journals in our field, and still today the two journals forming a category of their own and standing out from the other journals in our field as the, by far, most prestigious ones (Williams & Leatham, 2017)—they were typically, albeit not exclusively, of three kinds: *position and opinion* papers concerning mathematics as an educational subject, in general or in some of its aspects; papers offering an *exposition and/or discussion of a piece of mathematics* in an educational context (what the Germans call *Stoffdidaktik*); and papers presenting *teaching experiments*, sometimes but not always involving a control group, and typically accompanied by some quantitative data and an analysis thereof. The latter category was much more prevalent in JRME than in ESM. Here are a few examples to briefly illustrate the categories.

"On the ideal role of an industrial mathematician and its educational implications" in ESM 3(2), (Klamkin,1970), is an example of a position/opinion paper. The same is true of "On the Value of Status Studies in Education" (Scandura, 1971a) in JRME 2(2), which is a special issue devoted to (mostly critical) comments on the first IEA Study.

"Axiomatic approach to the teaching of solid geometry in grade IX" in ESM 3(2), (Roganovskij, 1971), and "Parts of a Systems Approach to the Development of a Unit in Probability and Statistics for the Elementary School" in JRME 1(4), (Shepler, 1970), are examples of mathematical exposition papers in an educational context.

Finally, "The Relative Effectiveness of Four Strategies for Teaching Mathematical Concepts" in JRME 1(2) (Rector & Henderson, 1970) and "A calculus-with-computer experiment" in ESM 3(1) (Smith, 1970) are examples of papers presenting teaching experiments and their outcomes.

Only seldom at that time did ESM and JRME publish papers of a conceptual or, even more rarely, a theory-oriented nature, such as the one found in Scandura (1971b).

The first two of the categories mentioned, position/opinion papers and mathematical exposition for teaching papers, have largely disappeared from today's research journals. They would hardly count as research contributions at all. Only the third category of papers, empirical investigations of teaching experiments, is alive and kicking also in 2018, even though modern papers of this category are very different in approach, structure and style from the ones published in the early days.

In the early days, empirical mathematics education research was largely inspired by psychology (cognitive, developmental and educational), especially the work of Jean Piaget, later also the work of the Soviet psychologists Vygotsky and Krutetskii. It would not be unfair to claim that psychology provided the conceptual and methodological umbrella for such research, together with the standard statistical testing apparatus employed in psychological and educational investigations, including psychometrics. This is also the reason why mathematics educators in those years assumed that the scientific basis for research had to be found in psychology, as reflected in the establishment of the PME in 1976, for some educators even to the point where mathematics education research was perceived as nothing but psychological research applied to the field of mathematics (e.g., Skemp, 1971). The notion of mathematics education research as a separate and independent domain, borrowing from a multitude of other fields, but subsumed under none, took several years to develop. I shall return to this issue below.

When teaching experiments and innovative curricula are placed on the agenda, the question of whether these experiments and curricula do in fact enhance learning becomes a crucial one. This gives rise to an even more fundamental question: how can we tell whether learning has improved? In the beginning this was gauged by means of test outcomes. If students taught according to approach A performed significantly better than students taught according to approach B, and it could be argued, with reference to some pre-tests, that students' performance at the outset was not significantly different in the two groups, then it was seen as safe to conclude that approach A was responsible for the difference in performance, and that, hence, approach A was more efficient than approach B.

This way of operating, however, placed all the responsibility of gauging learning on the test instruments employed, where it is taken for granted that test results are valid indicators of learning. But soon these test instruments themselves began to become subject to questioning and criticism. Many tests were seen as focusing too narrowly on routine procedures and factual knowledge, whilst paying far too little attention to conceptual understanding, problem solving, and mathematical reasoning, aspects of mathematical learning that are not easily detectable and measurable by usual timed tests, and certainly less so if these are of a multiple-choice format.

In other words, researchers from the 1980s on began to invest a massive interest in uncovering, understanding, characterising and analysing students' mathematical learning. This was further stimulated by surprising findings amongst various categories of students.

A classic example of this is the so-called Student-Professor (S/P) problem, which became the focus of lots of research in the 1980s and later. The problem, which was first introduced by Kaput and Clement (1979), and in greater detail by Clement, Lochhead and Monk (1981), reads as follows

> Write an equation for the following statement: 'There are six times as many students as professors at this university.' Use S for the number of students and P for the number of professors. (p. 288)

This problem was given to large numbers of students in different universities and in high schools as well, and it turned out that in general around two thirds of the students gave the erroneous answer 'P = 6S', instead of the correct answer 'S = 6P'. By the researchers who studied students' responses to the S/P problem, this error was termed 'the reversal error'. The fact that the reversal error was not only prevalent but also resilient, and largely immune to attempts to resolve it by way of teaching, made researchers want to understand the possible sources of the error. A variety of explanations were proposed, such as matching the sequence of symbols in the formula with the sequence of corresponding words in the task statement, confusing labels and number symbols, perceiving numbers as adjectives, and so on and so forth, but none of them seemed to be sufficient to fully explain the occurrence and prevalence of this error amongst large and diverse groups of students. The roots of the error were seen to lie deeply in students' cognitive models of the problem situation presented. Since the mid-1990s no new studies have been published so as to 'close the case', which therefore must be considered still open, at least to some degree.

Whilst it is obvious that factors of cognitive psychology are involved in the genesis of the reversal error, it is equally obvious that the nature of the error is also intrinsically of a mathematical nature, and hence cannot be exhaustively explained by psychology. In other words, more than psychology is needed for adequately studying the learning of mathematics.

Another insight gradually emerged amongst researchers. Quantitative studies are based on counting entities, or on measuring entities by way of some sort of scale. For this to make sense, it is essential to know exactly what it is that we are counting or measuring. Counting requires the presence of discrete objects, or entities resulting from a discretised continuum, such as student responses classified as belonging to different categories, types of errors made, categories of problem solving strategies chosen by students, types of mathematical arguments preferred by students, response categories in questionnaires, etc. If the demarcation lines between different groups, categories or types are ill-defined or blurred, then counting results based on entities being put into groups, categories or types, are neither valid nor reliable, because even slightly different decisions on where to place a given entity may result in widely different counting results. Similarly, when ordinal or numerical scales are used to capture student background variables, performance outcomes, levels of reasoning, percentages of correct answers in a test, etc., it is important that the measurements made according to the scale adopted are accurate and robust. If perturbations arise in the use of the scale, the outcomes of its use may well change dramatically.

Therefore, for quantitative methods to be put to valid and reliable use, it is crucial that the entities counted or measured are well-defined and well delineated. This is particularly true if quantitative comparisons are involved. It is of utmost importance to go beyond the surface of things. But this is fundamentally a qualitative issue. Therefore, this was one of the reasons why mathematics education researchers from the mid-1980's began to embrace, develop, and employ a wide range of qualitative methods, a trend which has gained momentum ever since.

Qualitative methods in mathematics education research generate their own issues. How can we document what we see in structured, semi-structured or unstructured interviews, classroom observations, or video-recordings, especially when the amounts of data typically are so immense that sample excerpts must be made, but on what grounds? How can the reader of a qualitative research study be convinced that such excerpts are fair, balanced and representative when it is unrealistic for the reader to check this against the entire pool of original data, even if this pool has been made available to interested readers?

The most significant issue is how qualitative data can be interpreted with reference to the initial aims of the research study, and to the questions posed in it, in such a way that this interpretation can generate justifiable and robust findings that can survive scrutiny with regard to possible alternative interpretations. Depending on the nature of the study, such interpretations often involve a variety of conceptual and methodological considerations drawn from diverse disciplines, mathematics included. Grounded theory (cf. Strauss & Corbin, 1990) is but one method of dealing with these challenges.

The key role of interpretation in qualitative studies generates the need for establishing some grounds for making interpretations. This is where theoretical frameworks, or even what some call theories, enter the stage. By invoking existing theoretical frameworks or theories or by putting forward new or modified theoretical frameworks, researchers hope to be able to provide a platform on which an interpretation can rely. It is interesting to note that some researchers attach a much more overarching and crucial role to theoretical frameworks. In the current description of 'Characteristics of a High Quality JRME Manuscript' (NCTM, 2018) one reads, under the heading of 'A Coherent Theoretical Framework':

- The study is guided by a theoretical framework that influences the study's design; its instrumentation, data collection, and data analysis; and the interpretation of its findings.
- The literature review connects to and supports the theoretical framework.
- Make it clear to the reader how the theoretical framework influenced decisions about the design and conduct of the study.

In other words, according to the editors of JRME, a high-quality research study is necessarily guided by and subsumed under the umbrella of a coherent theoretical framework. This implies that studies conducted in order to answer research questions that are not derived from and embedded in such a framework, but for which possible theoretical frameworks are to be chosen *post festum* in response to the research questions posed, as is also the case with the methods adopted to answer them, stand little chance of being published in JRME, at least in principle. A fundamental problem here is what counts as a theoretical framework or a theory. Looking into the ways these terms are used in the literature one sees that they are very far from being clear and well-defined. As a matter of fact, there is a plethora of meanings of these notions, and hence a plethora of uses of them, which in itself greatly weakens the capability of theoretical frameworks or theories to serve as the desired platform for the formulation of research problems, the design of a study or the interpretation of qualitative or quantitative data and findings. In a later section of this paper I shall give a more detailed analysis of the notions of theory and theoretical framework in mathematics education research.

Qualitative studies serve at least two different purposes. One purpose is to provide an existence proof of a certain phenomenon, and to offer explanations of the occurrence of the phenomenon, as was the case with the Student-Professor problem, without necessarily making any claim of universality. Another purpose is to pave the way for subsequent quantitative studies conducted to chart the prevalence, generalisability or scalability of the qualitative findings obtained, or to investigate interrelations, correlations, and ultimately causalities, amongst the phenomena and processes identified in the qualitative study.

Recent decades have seen a constant growth in qualitative studies with quantitative extensions, but qualitative studies continue to predominate in published mathematics education research.

So far, this brief survey of the development of research in mathematics education has focused on the nature of the *research approaches* adopted. From the 1990s on, psychology ceased to be the most significant discipline informing such research. Mathematics education research is no longer the cartesian product of psychology and mathematics. Instead, mathematics education research has developed an independent identity, or to be more precise, several independent identities, seeking inspiration from a wide array of disciplines: mathematics, statistics, computer science, psychology, cognitive science, neuroscience, philosophy, linguistics, semiotics, history, social science, political science, general education including curriculum studies, psychometrics, and so forth and so on.

Alongside this development, which represents a massive expansion of the network of links between mathematics education research and a multitude of other fields and disciplines, we have also witnessed a huge expansion of the domains, issues, questions, educational levels and target groups that mathematics education research sets out to deal with. Research has moved from focusing on primary and lower secondary mathematics education to also focus on pre-school and upper secondary mathematics education and on tertiary mathematics education at large, including mathematics as a service discipline as well as advanced university mathematics. Research has moved from being primarily interested in mathematics teaching to vesting a massive interest in mathematical learning. Research has moved from paying particular attention to concept formation and procedural skills to dealing with all aspects of mathematical work and activity, such as problem posing and problem solving, mathematical reasoning, mathematical exploration and formulation of hypotheses and conjectures, mathematical modelling, and the role and use of technology in the teaching and learning of mathematics, as well as mathematical beliefs and affects. It has moved from focusing on rules, algorithms and procedures to focusing on meaning, sense making and understanding in mathematics and on mathematical proficiency, competencies and practices. Research has moved from primarily focusing on pupils and students to also focusing on teachers, their education and professional development, their backgrounds, beliefs, and practices.

This enormous expansion of mathematics education research in several different dimensions (see Niss, 2007b) implies that, today, its nature is very multi-faceted and highly diverse. Yet, this is not well reflected in the dominant paradigms in research studies as published, especially, in our journals.

THE STRUCTURE OF MATHEMATICS EDUCATION RESEARCH ARTICLES

Of course, there is considerable variation in the structure and organisation of research studies published in journals and books. Nevertheless, without having been in a position to exhaustively chart and categorise the entire pool of research papers, I am willing to risk my skin by claiming that an *ideal-typical* (in the sense of Max Weber, 1904/49) journal paper of today can be characterised as follows below. To be sure, an ideal-type is meant to be a descriptive notion attempting to capture *what is* the case across a host of particular specimens, not a normative notion meant to prescribe *what ought to be* the case.

The study presented in such an ideal-typical paper is a small-scale, qualitative, empirical case study, oftentimes—but not always—accompanied by a quantitative survey of, say, item responses, student or classroom types, groupings, correlations between categories, etc., conducted within the boundaries of the cases involved in the study.

In the introductory section of the paper (which may or may not be labelled 'introduction') the theme of, the rationale for, and the background to the study are presented, and key literature pertinent to the theme is briefly reviewed so that the study reported in the paper gets situated in the existing research landscape. Depending on the underlying philosophy and nature of the study, one or a few research questions are then stated, possibly accompanied by comments on the purpose and goals of the study. Sometimes, the formulation of the research question(s) is postponed till or after the theoretical framework for the study has been presented. This happens when the research questions posed refer to and draw on concepts, terms and perspectives belonging to the theoretical framework(s) adopted, so that these questions cannot really be formulated outside the framework(s). At any rate, the outline of the theoretical framework(s), theoretical perspectives, or theoretical constructs constitutes a centre-piece of the paper. In that section the author introduces the key concepts and terms that are going to be used in the article and invokes the theoretical philosophy and perspectives meant to position and underpin the study. Next comes the section usually called ‘method(s)’ or ‘methodology’, but sometimes also ‘the study’, in which the approach to the empirical data generation and/or collection is presented, as is also the case with the statistical or other quantitative methods of analysis employed to analyse the data collected, provided quantitative issues form part of the study. Typically, the methods section also explains how the data generation or collection instruments were specifically put to use in the circumstances of the study, how analyses of the data were conducted, and how problems and challenges arising along the road have been resolved. The next section in the paper, typically called ‘results’ or ‘findings’, is devoted to stating the outcomes of the data collection and analysis and to interpreting these outcomes as answers to the initial research questions (provided these were stated explicitly). It is usually in this section that one finds—sometimes long, but seldom complete—excerpts of transcribed interviews, classroom dialogues, open-ended questionnaires and so on and so forth, which serve as empirical illustration and evidence of the findings. This section is followed by one or two sections containing a ‘discussion’ and presenting a ‘conclusion’, in which the author discusses the scope, validity and robustness of the findings obtained, identifies possible weaknesses and limitations, and offers an account of what new knowledge and insights concerning the teaching and learning of mathematics the study has produced, in addition to what was already known and understood prior to the publication of the study. Finally, many papers have a section devoted to speculations on possible educational or other implications of the study, or to pointing out future research perspectives, either in terms of new questions to investigate in continuation of the study or in terms of applications of the research methods employed in the study to new contexts or for new purposes.

Since this is an analytic reconstruction of the ideal-typical paper published in today’s front-line research journals, I do not claim that every published paper is structured exactly in the way just presented. The actual pool of papers does, of course, display deviations from this structure, especially when it comes to the relatively few purely theoretical papers. What I do claim is that the ideal-typical paper captures the essence of a highly predominant segment of published journal articles. I further claim that reviewers tend to strongly adhere to the template constituted by the ideal-typical paper and explicitly criticise papers that deviate from this template.

Theory, theoretical frameworks, and theoretical constructs

One of the key components in the ideal-typical journal article is the theoretical framework(s) and the theoretical constructs invoked to position and underpin the study. Elsewhere (Niss, 2007a) I have discussed the concept of *theory* in mathematics education and concluded that this is a highly ill-defined concept, which covers a wide variety of meanings, ranging from nothing but a limited set of singular notions and terms, as is very usual, over a more elaborate network of interrelated notions and distinctions, through to what I define as a theory (Niss, 2007a, p. 99):

- [A] theory consists of an *organised network of concepts* (including ideas, notions, distinctions, terms, etc.) *and claims* about some extensive domain, or a class of domains, of objects, situations and phenomena.
- In the theory, the *concepts are linked in a connected hierarchy* (oftentimes of a logical or proto-logical nature), in which a certain set of concepts, taken to be basic, are used as building blocks in the formation of the other concepts in the hierarchy.
- In the theory, the *claims are either* basic hypotheses, assumptions, or axioms, taken as *fundamental* (i.e. not subject to discussion within the boundaries of the theory itself), or statements obtained from the fundamental claims by means of *formal or material* (by 'material' we mean experiential or experimental) *derivation* (including reasoning).

This understanding of the concept of theory is only very rarely encountered in mathematics education research. Instead, what mostly passes as a theory is a much weaker construct in which one or more of the above three characteristics is relaxed, usually the hierarchical structure of the concepts, or the distinction between fundamental and derived claims. Many constructs termed 'theory' consist only of a few basic concepts and perhaps, but not necessarily, a few claims involving these concepts. Such a 'theory' amounts to putting forward some notions and terms, which in some cases might be called 'scientific term dropping', sometimes degenerating to 'scientific name dropping', that is, just naming the people who coined the terms in the first place but not going further than that.

The fact that 'theory' has turned out to be a rather vague and ill-defined notion may well be the explanation why most articles prefer to speak of *theoretical frameworks* or *perspectives* rather than of theories. A theoretical framework typically consists of an outline of a domain of entities, phenomena, or issues supposed to be captured by the framework, as well as a number of *constructs*, that is, a set of more or less connected concepts and terms by means of which it is possible to speak about the domain under consideration. Sometimes, a theoretical framework also includes certain basic but usually general claims about how to understand the entities and phenomena of the domain and their interrelations. Still, what counts as a theoretical framework, even though the notion is less ambitious than that of a theory, is highly diverse.

Whether we speak of 'theories' or, more modestly, of 'theoretical frameworks', these seem to serve at least six different, yet not contradictory, purposes (Niss, 2007a, p. 100): to provide *explanation* of some observed phenomena; to provide *prediction* of the possible occurrence of certain phenomena; to provide *guidance for action or behaviour*; to provide *a structured set of lenses* through which aspects or parts of the world can be observed, studied, analysed or interpreted; to provide a *safeguard against unscientific approaches*, or, differently put, against "ad hoc empiricism that is theoretically vacuous" (Schoenfeld, 1992, p. 181); and finally to provide *protection against attacks* from sceptical or hostile colleagues in other disciplines.

Frank Lester (2005) suggests to by and large abandon theoretical frameworks for research studies in mathematics education and replace them with what he calls *research frameworks* (p. 458):

> […] a research framework is a basic structure of the ideas (i.e. abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated.

He mentions three kinds of research frameworks, theoretical, practical, and conceptual, and perceives theory as but one specific kind of framework (p. 458). Lester identifies four advantages of using a research framework (p. 458): "A framework provides a structure for conceptualizing and designing research studies"; "There is no data without a framework to make sense of those data"; "A good framework allows us to transcend common sense"; "Need for deep understanding, not just 'for this' understanding."

Lester, in his article, joins forces with Gravemeijer (1994) and later Cobb (2007), when they make a plea for researchers to act as *bricoleurs*, "by adapting ideas from a range of theoretical sources to suit our goals" (p. 466). As long as we are so far away from having overarching, stable, and exhaustive theoretical frameworks to underpin all our research in mathematics education, I, for one, am delighted to position myself in the same camp as these distinguished researchers.

Research questions, methods and methodology

Another key component in the ideal-typical journal article is the research questions to which answers are sought. As indicated above, there are fundamentally two different roles and places of the research questions in an article. Either, which seldom happens in an ideal-typical article, they are stated at the beginning, after presentation of the background, before any mention of a theoretical framework, which means that the study is largely problem-driven (Schoenfeld, 1992, p. 180; Arcavi, 2000). Or, the research questions have been formulated within or in close association with the predominant theoretical framework adopted for the study, which means that the study is largely theory-driven (Schoenfeld, 1992, p. 180). The place and role of the research questions have a significant impact on the methods adopted to answer them.

In the rare case of an ideal-typical study for which the research questions are formulated outside a theoretical framework, the choice of an extant method or the design of a new one can either be made right from the beginning, aimed directly at dealing with the research questions posed, without being informed by any theoretical framework, or it can be made after theoretical frameworks have been chosen or established, in which case the methods at least have to be compatible with the frameworks adopted if not simply derived from or embedded in them. Of course, this is true also of the much more prevalent cases in which the research questions have been formulated within the chosen theoretical framework(s).

Looking at ideal-typical papers, irrespective of origin and position of the research questions inside or outside theoretical frameworks, the most significant challenge with respect to methodology is whether the methods adopted are actually suited to answer the research questions, entirely or in parts. More often than not, there is a marked discrepancy between the nature and scope of the answers sought and those actually obtained, which is often due to the inadequacy of the methods adopted.

CRITICAL ISSUES CONCERNING THE PREDOMINANT PARADIGMS IN RESEARCH ARTICLES

By insisting on explicitness and clarity of the exposition of its key components, the ideal-typical mathematics education research article has several merits. Also, it is not difficult to understand how it came about in the development of mathematics education research. It came about as an attempt to vaccinate the field against a number of actual or potential internal illnesses. One such illness is unspecific and imprecise language, in which notions and terms haven't been properly defined, if defined at all, but remain blurred and fuzzy. Another illness is lack of clarity concerning the purposes and aims of research studies, and of the questions they set out to answer. Yet another illness is sheer—sometimes idiosyncratic—'opinionating', substantiated by nothing but prejudices or impressionistic anecdotal evidence, possibly derived from the author's own convictions, experiences, or by experiments doomed to succeed simply because of the researcher's enthusiasm. Another illness is the inadequacy of the methods adopted to actually generate answers to the research questions posed. The final illness to be mentioned is a shaky theoretical or empirical basis on which inferences are made and conclusions drawn, implying that findings are less than satisfactorily supported. On top of all this comes the need for protecting our field against criticism from outside, which some believe is ascertained by defining, adopting, and implementing very rigid standards such as the ones represented by the ideal-typical research paper.

So, there are lots of good reasons for *discussing* criteria for quality in mathematics education research papers. However, there are serious problems in *establishing rigid standards* for such papers, like the ones described above.

The most important problem is that our field is very far indeed from having arrived at an agreed upon unifying foundation of mathematics education research, when it comes to the nature and place of significant issues and good research questions, to theories and theoretical frameworks to underpin and shape our research, to fruitful and effective research designs and accompanying methods that are conducive to providing relevant, valid, and reliable answers to the research questions, and when it comes to the place and role of general reflections on our field in research. It would be wonderful if we had such a unifying foundation of our field, but unfortunately, we do not. As Schoenfeld wrote in 1992 (Schoenfeld, 1992, pp. 179-180):

> In times of normal science, researchers have it easy. Established paradigms shape the vast majority of inquiries undertaken, and established methods appear to deal more or less adequately with the phenomena that are of paradigmatic interest. […] The learning sciences, also known as cognition-and-instruction or cognitive-science-and-education are decidedly not in a period of normal science.

This is no less true today than it was in 1992. Even though the ideal-typical research article is not, in any literal sense, the only one to be found in journals, it certainly gives the impression that mathematics education research is much more monolithic in its requirements than is in fact the case. This is highly unfortunate for at least two reasons. The fact that a rapidly increasing number of countries, universities, research agencies, and institutions only take articles published in journals into account implies that the researcher—and above all the novice researcher having recently entered the field—better ought to act according to the ideal-typical paper standards, especially when these are formulated by one of the two most prestigious journals, JRME, as characteristics of high quality papers. This clearly moulds research publications so as to fit the ideal-typical paradigm. The second, more important, reason is that eventually research that is conducted according to different perceptions of what is significant and relevant will gradually disappear because it cannot get published in the research journals, but at best in edited books. This constitutes a serious danger to our field. If it prematurely congeals into a narrow prototypical understanding of what counts as research, it will lose the ability to develop and renew itself. This is particularly true of theory-driven research (which "tends to focus attention on issues that are essentially within the scope of current theory" (Schoenfeld, 1992, p. 180)). If one of the main purposes of conducting mathematics education research is to pave the way for creating better theories that will eventually expand and consolidate our understand of the teaching and learning of mathematics in all its manifestations and under all its boundary conditions, which I believe it is, it is next to insane to force mathematics education research to be locked up in extant theoretical frameworks.

It is interesting to consider how old renowned articles would fare under the ideal-typical paradigm. In 2004, NCTM published a volume titled "Classics in Mathematics Education Research (Carpenter, Dossey, & Koehler, 2004). It reprinted 17 papers—with added perspectives from contemporary researchers—published between 1947 and 1996. Most of them would not stand a chance of being published today as a journal article within the ideal-typical paradigm outlined in this paper. They either lack a proper research question, a presentation of a theoretical framework, a presentation of the methods adopted, or a detailed discussion of the findings obtained, or of their scope and validity. This is rather telling of the unhealthy development in the publication paradigms, and hence of the underlying research paradigms.

Conceptually or theoretically oriented reflective research, without an empirical component, seems to be the kind of research that suffers the most from the predominant publication paradigm. As I see it, our field is simply not so far and well developed that we can afford to let this happen. Fortunately, the waters aren't completely frozen yet. Firstly, it is still possible to publish non-ideal-typical papers in our journals, also in the prestigious ones, even though strict space limitations apply to some of them (e.g., to FLM (For the Learning of Mathematics), and ESM, which allow no papers longer than 5000 and 8000 words, respectively). Secondly, a much wider spectrum of research papers can be published in books than in journals. It is often in book chapters that we find the most ground-breaking and innovative research contributions to our field. Can our journals really afford to let this be the case? However, even though the waters aren't completely frozen yet, we need to send ice-breakers out to keep the waters open for continued sailing also in the future.

WHAT CAN WE DO TO KEEP THE WATERS OPEN?

The most important thing for us as researchers, editors, reviewers, and supervisors, to work in order to keep the waters open is to insist on avoiding narrow, rigid templates for research studies and papers, as represented by the ideal-type of articles. Of course, I am not suggesting that this type of articles should be banned. As mentioned above, such articles certainly have their merits, but they should not stand alone. We must accept, foster, and adopt a multiplicity of approaches to all aspects of our studies and resulting papers, most importantly when it comes to the nature and role of research questions, theories and theoretical frameworks, methods and methodology, and interpretation of findings. As Kilpatrick (1993, p. 17) wrote:

> …researchers in mathematics education should never become wedded to a single approach, epistemology, paradigm, means of representation or method. All are partial and provisional, none can tell the whole story.

Abraham Arcavi (2000, p. 145) asked:

> Shall I choose a problem of interest (regardless of any theoretical/paradigmatical preconception), pursue it, and then try to shop around for frames which may help me to make sense of what I find?

and answered that such problem-driven research is certainly one viable option worth pursuing, not least for himself.

It is of particular importance that it is possible to publish non-empirical papers that focus on raising an issue, on putting forward new concepts and theoretical constructs, on proposing theoretical distinctions, on analysing, comparing or linking theoretical frameworks, or on presenting and analysing methods. It is also important that it is possible to publish papers on curriculum design and development, even without empirical evidence to corroborate them.

One unfortunate thing that the publish-or-perish pressure within academia has given rise to is that careful, constructive, and friendly scrutiny of already published papers is never made public. This may take place in doctoral courses or in the literature review section of PhD dissertations, but it is mostly not made a subject of public debate in our field, which is a pity. It is well known that reviewers only rarely are in a position to offer a thorough analysis of the submitted papers they have agreed to review. Reviewers work under time pressure and are neither paid nor really appraised for doing their job. But when reviewers are good they are the secret heroes of academia. To ensure critical but constructive academic debate we should consider ways of establishing fora in which published papers can be analysed and discussed in a non-hostile way. In this context, JRME has recently (Vol. 49(1), January 2018) taken the very positive step to place replication studies on the agenda of the journal. Such a step may also allow us to re-open abandoned tracks of past research, like the unsettled case of the reversal error in the Student-Professor problem and related problems referred to above.

Some—perhaps journal editors in particular—may well think that I have portrayed the state of affairs concerning publications in our field in a biased and too dire and pessimistic way, because they can point to many published papers that do not fall under the ideal-typical category of papers introduced above, papers that do in fact display some of the features I have made a plea for. Maybe so—I will definitely not claim to have fairly captured all sorts of published papers. However, I will maintain that I have captured a prevalent trend in research studies and publications that is detrimental and potentially dangerous to the development and future of our field.

In conclusion, the most important thing is that we work to keep the waters open rather than frozen, and that we remain/become open-minded, reflective, ready and able to discuss the nature of our field, also in journal articles.

This is not to say that we should now move into an era in which anything goes, an era in which our long-standing cumbersome attempts to generate quality criteria for research and publications are to be thrown into the dustbin. On the contrary, more than ever we must analyse and discuss quality, but in the same way as we should not endorse only one understanding and version of democracy, like the one found in, say, Switzerland, in the USA, or in Denmark, we should not allow too narrow and rigid paradigms to jeopardise our discussions or our field of research.

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AN AESTHETIC TURN IN MATHEMATICS EDUCATION

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Though often marginalised as simultaneously elitist and frivolous, the aesthetic remains a significant and understudied factor in the evolution of the discipline of mathematics, as well as in the learning and teaching of mathematics. One reason that the aesthetic often functions implicitly in school mathematics is that it conjoins two, often distinct, tendencies, between the rational and the sensible. By developing the political aspects of aesthetic practices, I propose an aesthetic approach to school mathematics that perturbs the prevailing values of order and clarity, and welcomes new values that energise students and teachers both in relation to the discipline of mathematics, but also in their experience of the world.

INTRODUCTION

In *The Evolution of Beauty*, Princeton ornithologist Richard O. Prum (2017) argues that evolutionary biologists have neglected one of the two evolutionary mechanisms hypothesised by Darwin. While the mechanism of natural selection, published in *The Origin of Species*, is well known and pervasively taken as confirmed by evidence, Darwin's other book—*The Descent of Man*—argues that the mechanism of aesthetic choice is necessary to account for the extraordinary ornamental diversity in the bird world, such as found in their plumage (the gorgeous features of the great Argus pheasant), their dances (the drumming of the ruffled grouse) and their mating displays (the avenue bower of the bowerbirds). Instead of seeing these instances of beauty as mere handmaids to natural selection—that is, that a beautiful plumage communicates the male's health, vigour and survivability—Prum (2017) mobilises extensive evidence to show that the two mechanisms work in tandem and that the arbitrary changes in ornamentations that occur in nature co-evolve with an aesthetic appreciation: "behind every biological ornament is an equally elaborate, co-evolved cognitive preference that has driven, shaped, and been shaped by the ornamental evolution" (p. 26).

If Darwin's purported understanding of this has been marginalised, it is because of the dangerous idea it entails: that birds could use their sensory and cognitive capacities to make decisions that are based on pleasure rather than utility, which violates theories of human exceptionalism. (And perhaps even more dangerous yet: the possibility of female sexual autonomy, which Prum illustrates well in his chapter on duck genital evolution.)

Prum's ideas are relevant to the topic at hand for two main reasons. The first points to the aesthetic turn currently at work at a broad intellectual level—even in the biological sciences—that is reclaiming earlier insights into the significance and force of sensory

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knowing and judgements of beauty and ugliness. The second is an important lesson in how science itself evolves: once neo-Darwinians homed in on natural selection as the mechanism in the theory of evolution—the prevailing hypothesis—all subsequent studies were thus conceived of as conforming to the hypothesis; everything is interpreted in terms of survival of the fittest. A similar tendency can be seen in mathematics education research, where any form of 'ornamentation' (beauty, elegance, ugliness, messiness, etc.) is relegated to a supporting role in a cognitive drama, and where cognition is declared to be the single mechanism that accounts for learning—a position that has been with us since Descartes and prevails even in theories of embodied cognition.

Let's take an example of this sort of discursive marginalisation in action. In a kindergarten classroom, children aged 4–5 years old are sitting on the carpet with a large screen in front of them projecting a blank dynamic geometry sketch. They have been asked what a triangle is. One boy gestures a triangle. They are asked whether they could also describe a triangle with words. One girl says, "it has three sides that… are connected". The segment tool is then used to construct three sides that are connected. Several children object that the shape that has been produced is not a triangle. Some say "it's upside down". The teacher protests: "you said a triangle was three sides that were connected. You didn't say it had to be right-side up." One boy says, "I don't like it." A silence ensues. Then one child turns his whole body so that he is looking at the triangle upside down. Two other children do the same. The teacher proposes to drag one vertex of the triangle around and the children are suddenly attentive, exclaiming, "ooh." One boy asserts, "It's actually a triangle." Another says, "You can stretch it out." A girl then states, "A triangle can be, um, a different shape but it just has three corners."

I have used this sequence in the past to celebrate the stunning definition for triangle—that a triangle can be any shape but just has three corners—that contains generalization and invariance (see Sinclair & Moss, 2012). But here I want to focus on an event that precedes this utterance, which was the statement "I don't like it". It is tempting to explain this statement away by supposing that the boy doesn't understand what a triangle really is, thus subordinating the aesthetic to the cognitive. But perhaps he wants to disrupt the conventions of geometry that dictate that all three-sided polygons should go by the same name. We do, after all, make special names for right, obtuse, isosceles and other types of triangles. Why wouldn't the downward-pointing triangle be perceptually worth distinguishing as well? Or perhaps he doesn't like it in the same way that mathematicians in the past have not liked incommensurable numbers, non-Euclidean geometries, infinitesimals, computer-assisted proofs and Ramanujan's infinite series for $1/\pi$.

This example invites us to ask what it would mean to treat aesthetics as a parallel 'explanation' in our theories of learning or in our approaches to pedagogy or to research. Naturally, the analogy I have offered with the bird world is limited, especially since the primary mode of aesthetic evolution in that world directly involves mate choice. But Prum (2017), and Darwin before him, would argue that plant and animal evolution in general changes through a carefully balanced joint process of utility and beauty. Indeed, using an example from the plant world, mathematics educator Rochelle Gutiérrez (2017) makes a similar point: “Brassica oleracea (Romanesco cauliflower) performs itself in both utilitarian (compact) and non-utilitarian (pleasing) ways that may get us to pay attention to its form and to continue to cultivate it” (p. 18). She envisions a school mathematics that similarly performs the useful/rational/true *and* the pleasing/emotional/beautiful.

Speaking of the aesthetic choices of birds would have been incomprehensible to Kant, whose foundational work upon aesthetics was unreservedly dependent on human response. His aesthetics was not a theory of objects or even art objects, but a theory of human perception and experience—where perception always played second fiddle to conception. Darwin then offers an expansion of aesthetic thinking both in terms of redressing the balance between the aesthetic and the cognitive and in terms of moving into the non-human world. Thinking of the aesthetic as a relation between objects is an interesting philosophical challenge, but it is also relevant to us for two reasons: (1) mathematical objects themselves are part of this non-human world and (2) thinking outside a human-centric position may have significant ethical, political, and environmental consequences.

It is perhaps for this reason that aesthetic matters have witnessed a strong return in philosophy over the past 15 years. This warrants the initial title of the talk, *An aesthetic turn in mathematics education*. One major figure in the contemporary re-thinking of aesthetics has been Jacques Rancière, who has pushed aesthetics outside of the philosophy of art and allied it with the political. I will describe some of his main ideas and then see how they operate in the context of mathematics and mathematics education. As I go along, I will weave in a few other contemporary takes on aesthetics including the work of Leonard Koren on wabi-sabi aesthetics, the recent writing of Rochelle Gutiérrez on mathematx, and the posthuman approaches to the aesthetic outlined by the Graham Harman. It is in this latter work that the idea of aesthetics as first philosophy—which could be a second title for my talk—has emerged.

THE AESTHETIC AS A COUPLING OF ART AND LIFE

I begin with the work of Jacques Rancière, a French philosopher, who is interested in the questions of how and when the aesthetic—as a mode of sensory knowing—operates politically. The connection between the aesthetic and the political arises when considering the relation between what can be sensed—what can be perceived, seen, felt—and what is taken as ‘common sense’, that is, as a social shared understanding of what makes sense. Art can function politically when it disrupts what is taken to make sense by changing the distribution of the sensible. In other words, Rancière (2004) gives a political twist to the notion of aesthetics as the science of perception by showing how sense perception structures and is structured that by which is perceived, which renders some things visible while rendering others invisible. That which is invisible does not count, has no value. For example, in the mathematics community, one can find periods of time during which visual forms of sense making were considered distracting and even dangerous, times during which being blind was seen as an advantage. That means that certain objects—such as diagrams—were invisible, thus privileging some social groups over others.

From Rancière's point of view, aesthetic practices are certain ways of 'doing and making' that have been chosen among many others. Once chosen, however, they generate particular forms of visibility and sensibility, and are thereby central to determining what others might call membership in a community of practice. Indeed, the practice of declaring that a theorem is beautiful is one way to indicate one's membership in the community of mathematicians. The school mathematics practice of having to 'explain your reasoning' generates a certain form of visibility and sensibility around what it means to know, and what is means to show that you know. Having to diagram or gesture your reasoning would change the distribution of the sensible.

Rancière (2004) asserts that the Western tradition of aesthetics confounds two oppositional concepts of sense—the first ('the art of the beautiful') associated with the autonomy of art and the second ('the art of living') with collective forms of sensibility. The word 'sense' describes, on the one hand, that which can be taken in by the eyes, ears, nose, etc., and, on the other, that which is seen as understandable or acceptable, as in the phrase 'it makes sense'. The autonomy of art refers to that which is free from the demands of functionality and explanation (a painting is only a painting if it is *not* useful). On the other hand, the painting is entirely reliant on sensory effect (a painting is only a painting if it is perceived). Thus, the aesthetic operates through the conjunction of sensing and of 'common' sense, conditioning our modes of perception, as well as our social institutions.

As de Freitas and Sinclair (2014) argue, a mathematical aesthetic operates through the same oppositional mix. It does so first by claiming that mathematics partakes of the autonomy of the aesthetic and then by insisting that one must live this aesthetic as a form of life. The autonomy is the purity of logical deduction (and its associated qualities of elegance, unity, efficiency) and the dependence arises from the embodied sensory engagement of the mathematician. The conjoining of the two is evident in the mathematics lecture performance notably discussed by Núñez (2006): the lecturer provides a formal verbal description of infinity, while at the same time 'performing' infinity by rhythmically gesturing a repeated, linear movement away from his body. While his speech negates his own being—through the formal voice of the mathematical discourse—his hand insists on carving out the iterative and manipulative space. He is at once de-temporalising, de-contextualising and de-personalising (Balacheff, 1988), *and* re-temporalising, re-contextualising, and re-personalising. At work in all of this is a figuration of infinity that grants autonomy to the aesthetic object and simultaneously takes it away.

The emphasis on pattern, symmetry, and regularity underscores the kind of autonomy that epitomises many accounts of the mathematical aesthetic. A mathematician might 'explain' a pattern with reference to the actions or operations that might be used to produce it, but this activity or labour does not engender the pattern. The emphasis on detecting patterns demands that the mathematician perceive or sense that which is independent of his or her labour, and must 'grasp as an idea' that which is autonomous—that is, she or he must internalise the autonomy and live the mathematical aesthetic as a form of life. The distinction between functionality and autonomy operates when a mathematical proof becomes aesthetic as it is granted a certain autonomy, that is, as it comes to be of no apparent everyday value. The determination of true or false is aligned with the functionality of mathematics, whereas being 'evident and compelling' is aligned with the aesthetic. This distinction construes mathematics as both autonomous (evident) and affective (compelling) in its aesthetic dimension. Indeed, we are compelled to submit to mathematics only when it achieves this aesthetic dimension, for it is only then that it truly embraces its autonomy. According to the mathematician Gian-Carlo Rota (1997), mathematical beauty is a way of sustaining an untouchable or unreachable sensory realm, for it is through aesthetic judgement that the ideal becomes real.

The aesthetic grants mathematics a sensory aspect, while simultaneously denying access to this encounter for all but a few. This can be seen in G. H. Hardy's (1940) discussion both of Euclid's proof of the infinity of primes and of the Pythagorean proof of the irrationality of √2. Mathematicians find these proofs beautiful because of their perceived simplicity and minimal amount of background knowledge required to understand them. Both of these judgements point to particular desires that fuel the mathematical aesthetic. On the one hand, simplicity involves a preference for truth to be present and singular, without difference or complication. On the other, minimal assumed knowledge relates to the desire that mathematics ultimately concerns pure reason rather than knowledge, because knowledge is tainted with the particularities of its historical (and sometimes geographic) context.

PROOFS BY CONTRADICTION

The proofs of the infinitude of primes and the irrationality of √2 are frequently used to exemplify the aesthetic qualities of mathematics. Interestingly, both are proofs by contradiction, which involves starting with an assertion that is the opposite of that which one aims to prove. Although not all such proofs are deemed beautiful, the act of beginning with the opposite claim is a highly aesthetic move, in that so doing enacts a kind of feigned indifference or autonomy with respect to the truth of the claim. This move immediately performs a kind of autonomy by setting up a distance between the aim of the proof and the manner of it. At the same time, the conclusion of the proof, in which the negation of the original assumption is proclaimed, can provide a feeling of closure, of wrapping things up harmoniously, which may bring a sense of pleasure or joy.

These two proofs also produce the unexpected, which is the object that was previously denied existence. In the case of the infinity of primes, yet another larger number is created, literally cobbled together from a collection of primes. We hear the speaker say, 'Do you see it now? You said it couldn't exist, but I have shown you one way to find one.' Another larger prime can always be generated from any finite set of primes; the new prime appears to come forth autonomously through logic, rather than through any act of calculation or imagination. In the case of $\sqrt{2}$, the creation of an irrational number plays havoc with the ontology of number and emerges, as though by magic. It defies our common sense, and we must recognise that there is a new way of delineating between the sensible (rational) and what was previously taken to be the non-sensible (irrational).

In writing about students' difficulties with proofs by contradiction, Leron (1985) highlights another aspect of the indifference that students might find troubling, which is the "mathematical destruction" (p. 323) at work when the proof asserts that the largest prime that has been posited does not exist. All that work to create this number $M = p_1p_2p_3\ldots p_n + 1$, a number which is either prime (and bigger than all the others) or has a prime factor that is not on the list); to make M something real, only to be forced to acknowledge no largest prime exists. If this delights mathematicians, Leron points out it only serves to frustrate most students.[1] He proposes a constructive approach in which the production of M and proving some results about it is separated from the negative assumption, which avoids the bewildering experience of having to destroy what had just been made while still retaining the formal effectiveness of the mathematical argument. Gutiérrez's (2017) mathematx might invite students to consider both constructive and "destructive" proofs, not only in terms of which is the more explanatory (Leron's concern) but which is the more harmonious. Might such an invitation, less focused on 'are you convinced?', allow students the space to find some joy in the strange logic of mathematical time? One proof could be for the explanation and another for the experience.

Indeed, it was really with the experience that Seymour Papert (1980) was concerned—more than with truth or explanation—when he provided undergraduate students with the first step of the proof of the irrationality of $\sqrt{2}$ (the setting out of the equation $p/q = \sqrt{2}$). When he asked them to generate transformations of the equation, they created the next step of the proof, which is to multiply both sides by q and then square them,[2] at which point Papert reports unmistakeable signs of pleasure which he attributed to their getting rid of both the fraction and the square root—each one of them being somewhat ugly because of their association with arduous simplification exercises. Or maybe it was the peek-a-boo effect in which q emerges out of the basement to become a main player in the equation. Papert was not asking the students to prove the theorem, or even to understand the proof of the theorem. He wasn't even asking them to simplify or solve the equation. He was asking them to do things with mathematical objects, in a way that aligns with the mathematician Wolfgang Krull's (1930/1987) sense, that "the primary goals of the mathematician are aesthetic, and not epistemological" (p. 49).

Incidentally, Papert (1980) highlights the important role that negatively valenced aesthetic responses can have in mathematical thinking. (The upside-down triangle was arguably another example.) It makes me wonder whether we might make more progress in our efforts to delight students with mathematics if we were prepared to admit that the beautiful tip of the iceberg rests on considerable ugliness.

I have been talking about how the mathematical aesthetic works in relation to Rancière's coupling of autonomy and dependence. I now turn to an example in mathema-tics education in relation to this coupling, and in order to explore another concept in Rancière's work, that of dissensus, which is related to the redistribution of the sensible.

DISSENSUS IN THE CLASSROOM

I now turn to the "Sean number" episode (which is described in Ball, 1993, who uses it to examine the dilemma of respecting children's mathematical thinking). The grade 2 class has been working with Ball as their mathematics teacher on patterns involving even and odd numbers when, one day, Sean announces that he had been thinking that "six could be both odd and even", because it was made of "three twos". The students discuss his proposal and dispute its legitimacy. He defends his assertion in the face of a growing concern on the part of the other students, including one student who rhythmically identifies the numbers as "even, odd, even, odd, even, odd, …" using a pointer on a visible number line above the blackboard—and thereby enacting the autonomy of the even/odd number pattern.

At this point in the lesson, the teacher draws six circles on the blackboard while asking, "are you saying that all numbers are odd then?" Sean uses these circles, dividing them into three groups of two, to "prove" to his classmates that six should also be odd. After working with the example 10, also involving partitioning of circles, Mei says, "like if you keep on going like that and you say that other numbers are odd and even, maybe we'll end it up with all numbers are odd and even. Then it won't make sense that all numbers should be odd and even, because if all numbers were odd and even, we wouldn't be even having this discussion!" Indeed, Mei's vision of what "make[s] sense" aligns with the conventional mathematical definitions of even and odd.

In contrast to Mei, Sean's contribution breaks with common number sense and offers an alternative way of organising the natural numbers in terms of factors. Indeed, Sean is indirectly distinguishing even numbers that contain at least one odd factor from those that do not, the latter having the specific label of 'powers of two' in mathematics. He is gesturing towards a tripartite division of numbers that consists of the really odd, the odd-and-even and the really even. Such a categorisation is entirely defensible mathematically and, indeed, in particular problem situations, perfectly functional. Sean disrupts a binary logic of even-or-odd.

Sean's contribution can be seen as an act of dissensus, in that it makes visible and audible what was invisible before—dissensus "enacts a different sharing of the sensible" (Rancière, 2004, p. 7). As de Freitas and Sinclair (2014) write, "[d]issensus is often a short-lived moment of dispute when the distribution of the sensible is contested, when someone stands, speaks out, touches an untouchable, eats a forbidden fruit, or gazes into a once-veiled object, a moment when the senses are used 'improperly' to dispute the equality of common sense" (p. 175). Whereas the consensus-making of Mei builds on a certain alignment between sense (as sensation) and sense (as meaning), dissensus is that which breaks up this alignment. Acts of dissensus shred the borders and divisions that currently partition the sensible. They are therefore political. In the context of the mathematical example offered earlier, an act of dissensus might have involved the publishing of a diagram in a mathematics journal, which would have challenged the prevailing consensus that diagrams were not sufficiently serious or rigorous.[3] In carving out of a new kind of number, which involves a different kind of seeing (of the factors), Sean's even-and-odd numbers also gain a certain independence—so that the cutting of 6 or 10 or 22 for that matter becomes a machinic process in which two sets of "unfair" numbers are newly detected. These even-and-odd numbers are both Sean's and not Sean's.

Rancière (2004) offers us two concepts that are useful for thinking the aesthetic in mathematics education. First, as highlighted in the Sean example, is the concept of *dissensus*. Like the example of the child who said, "I don't like it" about the upside-down triangle, dissensus provides a mechanism for becoming aware of the particular distributions of the sensible under which we operate as teachers and researchers and on the potential of new modes of sense perception that could matter in the classroom. The second concept is the conjunction of autonomy and dependence, which I want to think of in terms of *ambivalence*.

AN AESTHETIC OF AMBIVALENCE

I want to say a little more on this conjunction in relation to Rochelle Gutiérrez's (2017) "mathematx",[4] which is a mathematics that is both political and aesthetic. With mathematx, she aims to challenge current school mathematics practices that regulate students by:

> [privileging] algebra/calculus over geometry/topology/spatial reasoning; rule following over rule breaking; Western mathematics (culture free) over ethnomathematics (recognizing that even academic mathematicians are a culture); the "standard algorithm" over invented or international algorithms; abstraction over context ("just pretend this is real world"); mind over body; logic over intuition; and [by] encouraging students to "critique the reasoning of others" over appreciating their reasoning. (p. 22)

And she also envisions school mathematical practices that could enable students and teachers to think more productively about the current environmental crisis. Gutiérrez describes her vision as "a political statement about reclaiming the persons who have been lost when humans remain at the center" (p. 18). She mobilises the indigenous concept of *Nepantla*, which is the Nahuatl word used to describe an "interstitial space between worlds" (p. 12), where something is simultaneously neither and both. It is a space of possibility that is constantly changing. As she explains, "Nepantla can help us interrogate the idea that mathematics is both a universal endeavour and not a universal endeavour" (p. 14). She therefore wants to bring together a European mathematics, which aims to generalise, predict, and explain, with aesthetic aspects that are about pleasure: “Mathematx is a way of seeking, acknowledging, and creating patterns for the purpose of solving problems (e.g. survival) *and* experiencing joy” (p. 15; *emphasis added*). This resonates strongly with Darwin’s vision of evolution, but also echoes with the forces of autonomy and dependence that Rancière puts in opposition.

For Gutiérrez, the meaning of beauty and joy seems to be tightly connected to a sense of harmony that can be found in the respectful and reciprocal relations present in the more-than-human world. She offers the example of how the bean, corn, and squash “perform mathematics” by efficiently protecting each other as they grow, through careful spacing of leaves and intertwining of twines. The harmony is not just in the symmetry of the leaves of the corn or their perfect overlap, but in the way the corn, bean, and squash support each other, protect and are protected by each other. In the same vein, she asks us to consider how the mathematics we produce is being respectful, sustainable, and responsible? Gutiérrez is asking us to expect something different from mathematics, but also to change what we think is beautiful about mathematics. Indeed, what is deviant or ugly within the current school mathematics practice might be normal or beautiful in mathematx. And vice versa. Sean, for example, can be seen as engaging in mathematx, in that he is staying with the tension of wanting to keep the categories of even and odd, but also wanting to disrupt them, to seek a harmony of his own, in which some things can be this *and* that. For his teacher, the tension is not just about respecting his thinking, but also about being open to acts of dissensus that can function both aesthetically and politically in re-configuring what is taken to make sense and what is sensible, in order to open up opportunities for new knowledge.

One way to mobilise dissensus might be to engage in what Leonard Koren (2010) calls wabi-sabi aesthetics, which values imperfections, temporality, incompleteness, earthly crudeness, and contradiction (see also Maheux, 2016). Both Netz (2009) and Le Lionnais (1948) provide glimpses into how such values can occur in mathematics: the pathologies, asymmetries, confusions that are whitewashed by history or convention.[5] In the classroom, a wabi-sabi aesthetics might involve: a messy solution that nevertheless retains a trace of one’s problem-solving process; counting on your fingers; proposing a confusing definition. Dwelling on the contradictory nature of concepts such as square. It is a two-dimensional shape. Does it look like a frame, made up only of sides? Or does it look like a tile, including its interior? Is it perhaps both? In keeping with Gutiérrez’s invitation to encourage students to experience how plants and animals *do mathematics* in their own aesthetic way, there is yet another wabi-sabi opportunity that highlights the approximations and imperfections of the non-quite-symmetric snowflake or the not-perfectly-fractal tree.

In all of these examples, some form of dissensus is at work in that there is a redistribution of the senses, so that what makes sense is no longer what is clear and precise. But what would a mathematical classroom look like in this new regime? Instead of asking students to simplify algebraic expressions, might we ask them to complicate them? Instead of constructing equilateral triangles, might be ask them to construct triangles that are almost-equilateral or nowhere near equilateral? Instead of asking students to explain their reasoning, might we ask them to say something confusing about their solution?

In his essay 'Towards a pedagogy of confusion', Stephen Brown (1993) cites novelist Walker Brown who argues that there is a prevailing, societal "aesthetic unity" that makes it very difficult for people to live with their "fragmentary, haphazard and in-complete" lives. Stephen Brown does not directly blame school mathematics for this state of affairs, but does show how an aesthetic of clarity is seen as the 'natural' way that mathematics should be taught—even though it is anathema to the kind of confusion and conflict that actually characterises much of mathematical activity—and much of our lives.

Perhaps an aesthetics that is less about the beautiful, or even the ugly, might be in order. This might be an aesthetic of ambivalence. By 'ambivalence', I do not mean indifference or apathy; instead, I draw on the original German meaning, which refers to simultaneous conflicting feelings, where 'ambi' means both and around. Unlike ambiguity, which involves two mutually incompatible meanings, ambivalence mobilises more Nepantla indeterminacy and in-betweeness that evokes feelings and desires. It does not easily dwell on the continuum from beautiful to ugly.

What might such an aesthetic look like? Would it not be counter-productive to the values of consistency, connectedness, and coherence that is the very hallmark of mathematics? Stephen Brown (1993) provides an example of such an aesthetic at work, focusing on the productive potential of confusion. He asks us to imagine a teacher having clearly explained the concept of fraction (and of course, the students having found the explanation understandable). Then,

> a particularly enlightening activity might be to encourage the students either to find confusion in the category that appeared at first unproblematic, or to modify the concept so that it in fact acquires a healthy degree of confusions. (p. 119)

Can a fraction contain any kind of number? Can it look like this: 2.3; 0/8; 2/3/4? In shaking clarity in this way, Brown is asking students (and teachers) to live mathematics in a different way—as fragile, contingent. This mathematics can too often be covered up by images of progress and by the desire for purity. History is rich with such cover-ups and the pain they have cost, but little attention is paid to what has been lost in the march toward mathematical growth. Consider the growth of the number system. In the move to imaginary numbers, for example, we gain a way of solving certain kinds of equations containing the square roots of negative quantities, but we lose the orderedness of the numbers we had before. Has it been worth it?

Might an aesthetic of ambivalence, which acknowledges such confusion and loss in mathematics, help attune learners to a more complex, interdependent, and compromised life? I think it was this kind of mathematics that mathematicians and philosopher Alfred North Whitehead (1948) had in mind when he lectured on the topic of "mathematics and the good" and argued that mathematics is uniquely well positioned to enable humans to appreciate contingency, and thus avoid dogmatism and arrogance. It may indeed be arrogance that is responsible for the long-standing regime of human-centrism, where what it means to think, to know, to feel, to be conscious, to prefer, to love, to be, is defined in strictly human terms.

The recent posthuman turn asks us to consider the different differences there might be if non-human things *also* can be, can know, can prefer. Gutiérrez (2017), for example, evokes how animals and plants *do* mathematics. Elizabeth de Freitas and I (2014) have also explored how a posthuman, materialist perspective might shift our conception of mathematics itself. In our inclusive materialism, mathematics is neither just socially constructed nor idealist or transcendental: mathematics partakes of the material world and has a life of its own—like the rock, the hammer, the tree, and the ocean—that is not just handed down from some Platonic realm or some God, and not just a human or social construction. In the next section, I use the work of Graham Harman (2007) to explore what a posthuman aesthetics might look like, and how it might matter in mathematics education.

A POSTHUMAN AESTHETIC

A posthuman aesthetic would break with the Kantian aesthetics that is determined by its relations to human perception. In Prum's (2017) work on evolution, there is already a non-human aesthetic at play in that birds—and other animals—can have preferences, can make choices based on beauty, can perceive things in the world in their own way, and not necessarily in the way a human might. A posthuman perspective invites us to extend this way of thinking not just to animals but also to plants, to sand, to hammers, to mathematical objects.

For Graham Harman, a speculative realist, such a posthuman perspective requires aesthetics, since aesthetics is precisely about how objects interact with each other, how they sense one another, move towards each other, hear, touch, crash into each other. Harman (2007) thus argues for "aesthetics as first philosophy" (not ethics, nor metaphysics) inasmuch as aesthetics is about how "individual substances interact in their proximity to one another" (para. 23)—where those individuals are not only humans, and not only animals, but also non-human beings such as "snowflakes [that] rustle the needles of the quivering pine" (ibid.). For Harman, this interaction occurs through a proximal relation in which one object *affects* another, where one object "*allude*[s] to the reality of the other" (ibid.). Allure thus becomes a phenomenon of causal relations.

But allure, in belonging to the world, also occupies its multiple scales, including the individual scale of human senses and the micro-phenomenological scale of what Whitehead (1967) might term pre-individual "vibration"—a concept he used to describe the agency of the environment itself. In that environment, the child who states his dislike of an upside-down triangle is entangled in an atmospheric complex of heat, light, and pressure jumbled together with his fingers, a screen, a carpet, his classmates and teacher. The triangle is really there, and not just the sum of his perceptions and conceptions of it. When he looks at it, he sees some of its qualities, but he can never know or sense the triangle as a whole; in other words, the triangle exceeds all possible human relations. This is not to say that the triangle can think and feel. But between the triangle and the boy, it is through allure that the boy can say, “I don’t like it,” thereby generating a new relation between himself and the triangle. As Harman (2005) writes “[a]llure splits an object from its sensory notes” (p. 245). Allure refers to a new kind of sensory knowing, to a sensory knowing that is not restricted to the human senses, and may well operate within a pre-affective, pre-conceptual kind of intimacy.

In a similar vein, that is, within a posthuman perspective, Petra Mikulan and I propose we move away from a human-centric mathematics that always seeks to explain, control and unify in human terms (Mikulan & Sinclair, 2017). This mathematics would be taken as a force in its own right, not always inscribed in the social constructivist logic of human relation. In the same way that we can imagine all the imperceptible or barely perceptible fields before or beyond the point of view of the human, when it comes to the more-than-human world (the high-frequency sounds that dogs can hear, the forces of subatomic particles, the memory of migrating birds), what if we could do the same for mathematics? Such a mathematics could never be completely known, completely tamed, or “covered” because it always exceeds human and even more-than-human relations. We can think, for example, of the relations the circle has with the puffer fish, who creates a perfect circle in the sand with his fin (never swimming up to look at what has been carved out from above). A mathematics with a force of its own necessarily has some autonomy, some non-sense (from the human point of view).

Imagine a triangle constructed in a dynamic geometry environment. Perhaps it looks upright and equilateral at first, when the sides are joined. But then the vertex on top starts to move around, first transforming the triangle into something short and stout and then into something extremely long and skinny. At a certain moment, the vertex lands *on* the opposite side. What is this shape that has been produced by this random movements of a vertex? Is the child who insists it is still a triangle speaking non-sense? What about the child who insists it is not because it has no inside? Perhaps it is both a triangle *and* not a triangle. Is it a singularity that must be explained away (the teacher might say: ‘don’t put the vertex there’ or ‘that doesn’t matter’)? Indeed, this situation is safely ignored in pencil-and-paper environments where such an event cannot occur[6]. But allowing it to emerge not only alerts students to the material force of mathematics, to the desire of the triangle to explore all of its possible configurations and to collapse, like whales beaching themselves to die, but also to aesthetic tendencies that resist the temptation to explain and ignore.

Perhaps it is in embracing chance encounters, singularities, and ambiguities that we can come closer to the “power of imaging that is not oriented to the eye of recognition, the eye that views the world according to its own already organized desires” (Colebrook, 2014, p. 77). Might such a more-than-human aesthetic in school mathematics offer an alternative way of living, one that complements Gutiérrez’s vision of mathematx, where a better attunement to the more-than-human world and to the particular relations forged in this world?

Harman (2005, 2007) and Rancière (2004) may seem worlds apart, but their work can both be seen as exploring and exploding the conjunction *and*. For Rancière, the aesthetic is about autonomy *and* dependence; for Harman, it is about touching *and* not touching, communicating through proximity *and* emerging as an independent reality. For her part, Gutiérrez shares with Rancière a desire to dwell in the interstitial space of mathematics as a universal endeavour *and* not a universal endeavour; of utility and ornament; prediction and harmony. And she shares with Harman a desire to challenge human exceptionalism, but unlike Harman, is motivated primarily by a political project, which brings us back to Rancière. As is evident in the history of western civilisation, we have a tendency to turn *ands* into *ors*, that is, to resolve into binaries and exclusiveness. An aesthetic philosophy for mathematics education could provide an alternative: a way to sustain more inclusive coordinating conjunctions and to better account for the often accidental, barely perceptible allure through which new relations with mathematics can be generated.

CONCLUSION

As Leonard Koren (2010) makes clear in his book *Which 'aesthetics' do you mean: Ten definitions*, the word 'aesthetic' can mean very different things. It can mark a certain style of doing things (like in art or architecture). It can be associated with that which is beautiful or elegant. It can describe the qualities of certain objects (paintings, proofs) or the nature of certain types of experiences. In the context of mathematics, most writings on the aesthetic have focussed on the qualities that have been valued by mathematicians, such as elegance, interest, and surprise (i.e., Hardy, 1940; Inglis & Aberdein, 2015; Raman-Sundström & Öhman, 2013; Thomas, 2016; Wells, 1988). One thrust of mathematics education research has focussed on the extent to which students share similar aesthetic preferences to those of mathematicians (Dreyfus & Eisenberg, 1986; Marmur & Koichu, 2016). Another thrust, which also stays close to the dominant aesthetic values in mathematics, focuses on how students can use these aesthetic qualities in their own mathematical activity (Eberle, 2014; Dietiker, 2016; Scott, 2018; Sinclair, 2001) or on how teachers might use them (Crespo & Sinclair, 2008).

In this paper, I have been exploring different ways of thinking of the aesthetic and its relevance to mathematics education, some of which might feel uncomfortable either because they move away from our desire, as mathematics educators, to share in the delight that can be mathematics, or because they challenge our own privileged role in the world. However, I am not arguing for an aesthetic of the ugly or an aesthetic of ambivalence or even a posthuman aesthetic to take over the possibility of also experiencing beauty, certainty, or security in the mathematics classroom. Instead, I am offering new approaches that, in different ways, challenge current regimes that either anaesthetise teachers and students or privilege certain distributions of the senses that may have important social and political—and even environmental—consequences.

Finally, mathematician René Thom is often quoted as saying that, even without knowing it, all teachers have a philosophy of mathematics. Similarly, as is currently being documented across a variety of disciplines in the sciences and social sciences, humans are fundamentally aesthetic creatures (see Chatterjee, 2013; Dissanayake, 1992). Therefore, teachers, even without being aware of it, also have an aesthetic of mathematics. How does this aesthetic shape their students' experiences? How are we acknowledging this aesthetic in our research in mathematics education? What is being overlooked, undervalued, misinterpreted? What consequences are there for our lives, for the planet, for the health of mathematics?

Notes

[1] Indeed, not every mathematician accepts such a "non-constructive" proof as legitimate (and finding a constructive proof can stump even a Fields medallist; see Gowers, 2010).

[2] There is a Leron-like constructive separation possible here too, namely proving that all prime factors of the square of a whole number come in pairs. Then simply looking at $p^2 = 2q^2$ reveals its impossibility.

[3] A consensus in mathematics relates to the requirement that the truth of the proof be verified by a human, which meant that part of the proof of Kepler's sphere-packing conjecture could not be published in the *Annals of Mathematics* (Holden, 2003). Dissensus may arise in the redistribution of the sensible when computers too can verify truth.

[4] Gutiérrez chooses the 'x' in 'mathematx' (pronounced 'mathematesh') in part to evoke the x used in latinx, which is an ending that does not privilege the binary assumptions of the male/female words latino/latina.

[5] Lakatos (1976) provides a good example of the convention in which all the messiness of a result such as $V - E + F = 2$ becomes hidden in the definitions.

[6] Of course, technology is not required. Zazkis (1998), for example, shows how a similar kind of ambivalence arises in the concepts of divisor and quotient.

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PLENARY PANEL

CHICKEN-EGG CYCLES: WHAT NEEDS TO COME FIRST, HIGH PERFORMANCE OR POSITIVE AFFECTIVE VARIABLES REGARDING MATHEMATICS?

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> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated. You can see exactly where you were. (Andrew Wiles, from Nova (1993))

The metaphorical narrative above, though expressed as a highly individualised experience, motivated me to begin with the underlying affective components of the personal mathematical experience described.

Reflections on descriptions such as this, expressing a strong relationship between affect and cognition regarding mathematics, are not a novelty in our research field. They indirectly supported mathematics educators' interests in mathematics-related affect in problem-solving research in the eighties (see Zan et al. 2006); and they have recently been revisited to investigate the nature of illumination in mathematics, which includes, but cannot be reduced to, a single moment of pure insight (Liljedahl, 2013).

The images expressed in Wiles' narrative are striking in their suggestion that no human activity is free from hard work, embedded in 'the six months or so' of the entire process; and especially striking through the suggestion of delight after 'finding the light switch and turning it on' by just seeing 'where we were' – a situated-affect manifestation, surrounded by the particular context he described. Although indicating the existence of a hidden mathematical structure 'out there', represented by mansions and furniture, light switches are not referred to as gifts, but are 'found' through acting and experiencing; and individual's agency is required to switch them on.

Those thoughts came back to me while wondering about range of meaning for delight in mathematics education, the theme of this conference. As I am not a native speaker, I naturally started by asking myself what delight could broadly mean in the first place. In the Merriam-Webster dictionary, for instance, the origin and etymology of the word ascribes charm, fascination, dazzle, related to the 'Middle English deliten, from Anglo-French deliter, from Latin delectare, frequentative of delicere to allure, from de- + lacere to allure'. Then, I learned that the current word delight was written delite as late as in 1590, and that it is an – erroneous – spelling of after light, being already used earlier in the 16th c. This is how Wiles' narrative came back to me, as the reversal of a (mis-)spelling that resulted in delight – the emancipation of the after light effect as

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 69-75). Umeå, Sweden: PME.

simple, though not naive, positive feelings regarding mathematics and as related to a (high) performance in mathematical activity, interwoven with an 'awareness of where we were'.

Mathematical experience in mathematics education has, without doubt, other specificities and requirements untouched by the above narrative. However, regarding mathematics-related affect and its relation to performance, I learned from it that, in essence, an awareness of where we are could be curiously a representation of delight, with fuzzy boundaries, and (high) performance or the achievement of finding light switches and turning them on. We will return to this point later.

Firstly, I will depict from the narrative some pioneering foci of research on mathematics-related affect, such as 'mathematics anxiety' and 'attitude', the two constructs borrowed from social psychology in the 60's and 70's (see Zan et al., 2006). It seems reasonable to consider that experiencing a mathematical activity in terms of entering a dark mansion would touch affective variables such as anxiety and attitude. In both cases, one may wonder:

> Why do some people cope with the mathematical experience, have fun and succeed, and others don't?

Most researchers in those earlier decades addressed questions such as the one above through searching for causal correlations between anxiety and performance, and between attitude and performance (see, for example, Di Martino & Zan, 2015). While there were those who considered test anxiety to be a result of repeated unsuccessful experiences, or, in other words, that low performance precedes negative feelings related to mathematics, the main assumption was, on the contrary, that "test anxiety inhibits cognitive processes, e.g. recall of prior learning, thereby reducing performance" (Zan et al., 2006, p. 2).

This assumption seems to offer a reasonable explanation for why some people cope with the mathematics and others don't.

By the same token, research on attitude was based on beliefs about the relationship between affect and achievement, and considered that "improving affect would improve performance" (Zan et al., 2006, p. 3).

In the late eighties, many mathematics educators claimed that the causal relations between attitude and achievement were unclear, as well as the very notion of attitude. And, in the nineties, the meta-analysis in Ma and Kishor (1997), adopting a common and recurrent definition of attitude in terms of positive or negative feelings associated with mathematics (Di Martino, 2016), confirmed a causal relationship from positive feelings to achievement, "but the effect size (.08) was too small for practical relevance" (Zan et al., 2006, p. 4).

Less than a decade after, Hannula et al. (2004) found indications that mathematics-related affective variables can be seen as indicative of learning outcomes or as predictive of future success. These results raised again the earlier queries on the relationships between performance and affect through the question:

> Do positive feelings (or positive affective variables) regarding mathematics precede high performance?

On the other hand, which 'positive feelings' or 'affective variables' are we referring to, and in which sense or senses do they precede performance? For Zan et al. (2006), a theoretical model of the relationship between the various constructs and the comparisons between the different studies were difficult to develop until the late seventies, due to the need for clarifying their theoretical foundations. In the eighties, the research on problem solving was profitable ground for theoretical development, stimulated by the relationships between cognitive, metacognitive and mathematics-related affect, expressed in mathematicians' narratives of their professional activity, and by mathematics educators' perceptions of failure in the problem solving of students who apparently had the cognitive resources at need. Expansions in affect variables included the notions of *beliefs*, *attitudes*, *emotions* (McLeod, 1992) and *values* (Bishop, 1988; DeBellis & Goldin, 1997).

In addition, the important shifts in methodological perspectives in mathematics education and the emergence of new research paradigms represented a move from the causal-relationship paradigm to more interpretative ones (Zan et al., 2006; Di Martino, 2016).

For instance, attitude "is now considered to be an interpretative instrument to understand the reasons for intentional actions: intentional actions involve complex relationships between affective and cognitive aspects" (Di Martino, 2016, p. 3).

The attitude construct is described in Di Martino and Zan (2011) as a three-dimensional model, which "acts as a bridge between beliefs and emotions, in that it explicitly takes into account beliefs (about self and mathematics) and emotions, and also the interplay between them" (ibid, p. 479). Interestingly, teachers themselves often relate "negative attitude" with students' failure "as a claim of surrender rather than a precise diagnosis to activate a didactical intervention" (Di Martino, 2016, p. 6). In this sense, one would argue, based on classroom pedagogical experiences, that negative affective variables precede failure, offering mathematics-related attitude as an example. On the contrary, recent research on students' self-efficacy beliefs considers that, "in educational settings, previous success develops self-efficacy while failure undermines it" (Pantziara, 2016, p. 7). In fact, results from a longitudinal study in Finland (Hannula et al., 2014) indicate that "mathematics achievement and self-efficacy have a reciprocal relation, where the dominant effect is from achievement to self-efficacy" (p. 249). Does that mean that positive feelings, or more broadly, positive affective variables, are mainly dependent on having (many) positive mathematical experiences? In other words:

> Does (high) performance precede positive mathematics-related affective variables?

At this point, and in looking for alternatives to approach the affect and performance relationship, I turn back to Wiles' narrative. There, the delight of the *awareness of where we were* sounds like a long-term affect, or consciousness after an 'intense illumination leap'. According to the narrative, the latter follows long-term engagement in a mathematical experience which leads to the discovery of the light switch and to the conscious act of turning the light switches on. Liljedahl (2013) concludes that "illumination in mathematics is in the affective domain" (ibid, p. 264) with intensity "regulated by the intensity of the preceding conscious work" (ibid, p. 263) in which an individual is engaged. Thus, a slightly different and complementary question may be raised:

> What kind of mathematics-related affect and cognitive structure supports a challenging long-term mathematical experience engagement?

McLeod (1992) included a description of affective constructs along dimensions of decreasing stability and increasing intensity – the emotions being the least stable and most intense affective variable. More recently, Midleton, Jansen and Goldin (2016) refer to similar dimensions as "in-the-moment" or local affect and "long-term" traits. For these authors, staying engaged in the activity seems to be related to tensions between the in-the-moment and long-term patterns of task engagement. On the other hand, structures of engagement are relatively stable, and they interrelate intrinsic, extrinsic, individual and social factors. Central to the self-regulation of engagement is motivation, "the reason we engage in any pursuit, mathematical or otherwise" (Middleton, Jansen & Goldin, 2016, p. 18). The notion of motivation constitutes, with emotions and beliefs, three broad categories of affect (Hannula, 2012). It is related to an overall evaluation of the activity by an individual determining "the value the task has, to choose whether or not to engage, and to plan their course of action" (ibid, p. 19). From this perspective, motivation is related to affective resources, and to *productive learning behaviour*, or performance, understood as

> "a social construct formed from the interaction of learner personal learning states and mathematical dispositions, their home community, their classroom or learning environment community, and macro-cultural constraints such as curriculum, assessment, and cultural attitudes" (ibid, p. 18).

Adopting Middleton, Jansen and Goldin's (2016) perspective, we re-interpret the question to be addressed as:

> What needs to come first, motivation or high performance regarding mathematics?

Two teams of researchers are approaching this question through different theoretical lenses.

Wim Van Dooren's research on problem solving and problem posing has related students' performance to their beliefs on what problem solving in a math classroom involves (see Chen et al., 2015; Dewolf, Van Dooren, & Verschaffel, 2011). Cultural approaches and underlying values in effective mathematics teaching and learning have been addressed by Qiaoping Zhang's research (see, for example, Zhang, 2014; Zhang, Barkatsas, Law, Leu, Seah & Wong, 2016). They can make a case for motivation preceding high performance, and are followed by another team of researchers, Wee Tiong Seah and Francesca Morselli, who assert the opposite.

Wee Tiong Seah's main research interests are on culturally situated notions of values and valuing (see, for example, Seah, 2018; Seah & Wong, 2012) that are learnt and embraced by learners, teachers and the wider education community, while Morselli is concerned with the intertwining between affect and cognition in problem solving and proving (see, for example, Morselli & Furinghetti, 2007; Furinghetti & Morselli, 2009).

These partnerships have the potential to magnify those earlier researchers' theoretical lenses in the debate on the relation between affect and performance.

FORM AND STRUCTURE OF THE PANEL

The panel will be held according to the Oxford-Style debate structure, where two debate teams of researchers will express different positions about the relation between affect and performance by somehow addressing the question of what needs to come first, positive feelings regarding mathematics or high performance. In order to encourage participation and engagement during the panel, the proposal is to have a "polleverywhere" (www.polleverywhere.com) when the session starts, with a slide open on the screen allowing participants to vote on responses to the question:

> Does motivation regarding mathematics precede high performance, or does it follow high performance?

Following Forgasz' (2015) organization for the panel, each speaker will have 10 minutes to present their perspectives. The chairperson will outline a rationale for the topic, present the two debate teams and describe the rules of the debate. The opening speaker will sustain that motivation precedes high performance and is in charge of presenting the definitions of the technical terms that will be used. Next, the first speaker for the second team will begin with a brief rebuttal for the arguments presented by the first speaker and comment on the definitions used when appropriate. The process will continue until the last speaker's presentation.

The debate will be closed with a five-minute presentation by one speaker of each team. After all the presentations, there will be ten minutes for comments on some of these questions to be taken by each debate team (5 minutes each). We conclude with ten minutes for questions from the floor.

Participants will be asked to take a vote, again, as by this time they may have changed their views.

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WHY MOTIVATION NECESSARILY PRECEDES HIGH MATHEMATICAL PERFORMANCE – BUT NOT VICE VERSA

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Motivation provides a learner's impetus to initiate learning and later the driving force to sustain the learning. Through the analysis of the motivation concept, empirical studies and related research methodology, we conclude that positive motivations precede high mathematical performance. More experimental evidence seems desirable to understand how affective factors influence students' achievement.

INTRODUCTION

Outsiders often consider mathematics to be a rational activity in which logical deduction determines the flow of thinking and problem solving, and in which affective factors play no role (Törner, 2014). However, learners experience a wide range of feelings and moods in relation to learning mathematics, to problem solving, to performing on a test, and so on. Various terms are used to denote the diversity in affective variables. One major dimension that is used to distinguish affective variables is the temporal one (McLeod, 1992). Certain affects can be described as emotions: rapidly changing states of feelings that are directly related to specific activities and experiences. Other affective variables are considered to be longer lasting and moderately stable, including attitudes that learners have regarding mathematics as a school subject, and beliefs about mathematics as a scientific discipline (Phillip, 2007).

As Zan, Brown, Evans, and Hannula (2006) argue, there are two major argumentations to conduct research on affective variables regarding mathematics. First of all, positive affect regarding mathematics is considered to be related to a better mathematics achievement. This will be the main focus of the current paper, and we will provide theoretical and empirical arguments to substantiate this idea. Second, some consider positive affect regarding mathematics as significant *per se*. Indeed, in various mathematics curricula around the world, it is a common desirable goal that learners develop positive attitudes and beliefs about mathematics. While this goal seems to be an intrinsic one, it serves a purpose in the longer run: If during their education learners have acquired a positive affect regarding mathematics, they will also in the future (be it in a school setting, in daily life, or in professional circumstances) be inclined to practice more, undertake more challenging tasks, be more persistent when solving problems,

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etcetera. This is already a first argument to make the point that was made in our title: high mathematical performance is preceded by positive affect.

FOCUS ON MOTIVATION

The term "positive affect" is very general. To make it more specific, we have decided to focus the panel discussion on the construct of motivation. In line with Ames (1992), we would define motivations as reasons individuals have for behaving in a given manner in a given situation. They exist as part of one's goal structures, one's beliefs about what is important, and they determine whether or not one will engage in a given pursuit. As Hannula (2006) argues, motivation can be conceptualised as a potential to direct behaviour through the mechanisms that control emotion. In that sense, it is related to other affective variables as it regulates them.

Motivation is not a unidimensional construct in the sense that it only varies in intensity (learners can just be more or less motivated). The self-determination theory (Ryan & Deci, 2000) focuses on major qualitative differences in the way in which learners can be motivated. While nowadays these differences are formulated in a subtler way, the major distinction is that between intrinsic and extrinsic motivation. Intrinsic motivation is the drive or desire of the student to engage in learning "for its own sake". Students who are intrinsically motivated engage in academic tasks because they enjoy them. They feel that learning is important with respect to their self-image, and they seek out learning activities for the sheer joy of learning (Middleton, 1995). Their motivation tends to focus on learning goals such as understanding and mastery of mathematical concepts. Students who are extrinsically motivated engage in academic tasks to obtain rewards (e.g., good grades, approval) or to avoid punishment (e.g., bad grades, disapproval). These students' motivation tends to centre on such performance goals as obtaining favourable judgments of their competence from teachers, parents, and peers or avoiding negative judgments of their competence (Ames, 1992).

This distinction between types of motivation is an important one, as different types of motivation lead learners to do different things. Lepper (1998), for instance, has shown that learners who are motivated intrinsically exhibit a behaviour that can be considered as pedagogically desirable, such as showing an increased time on task, a persistence in face of failure, a more elaborative processing and monitoring of comprehension, a selection of more difficult tasks, greater creativity and risk taking, etcetera. The link with mathematical performance seems obvious and it may also have different effects on mathematics learning in different cultures (Zhu & Leung, 2011). We would argue that mathematics educators would generally agree that when learners exhibit activities associated with intrinsic motivation, they will learn more and in a deeper way, and as such show higher mathematical performance.

CONCEPTUAL ARGUMENTATION

So far, we have argued that intrinsic motivation would lead to a higher mathematical performance than extrinsic motivation. However, one can argue further that extrinsic motivation is still better for performance than no motivation at all, and so, more generally, that motivation–of any kind–precedes high performance. Our claim is that a high motivation *necessarily* precedes a high mathematical performance, while the opposite (motivation following high performance) is not necessarily the case.

Our first argument is a theoretical one, based on simple logic. While there are various definitions of motivation available in the literature, they all let motivation precede in time before any mathematical performance. We just take some excerpts from these definitions to make our point: “the reason we engage in any pursuit, mathematical or otherwise” (Middleton, Jansen, & Goldin, 2016, p. 18), “determine whether or not one will engage in a given pursuit” (Ames, 1992), “reasons individuals have for behaving in a manner in a given situation” (Middleton & Spanias, 1999, p. 66).

An essential point in each of these definitions is that motivation precedes performance in time, and that it is causally related to that performance. Learners will not show any behaviour in the total absence of motivation, so motivation *necessarily* precedes performance. Importantly, the opposite is *not* necessarily the case: A high motivation for mathematics may also occur after a learner did not perform well at all. For instance, the learner may, for some reason, not be aware of the actual quality of his or her performance and believe it was excellent. This may motivate him for the future.

A second aspect in the definitions of motivation deserves attention too: Even if we would accept the idea that motivation may follow after a good mathematical performance, this motivation is still oriented on a *future* mathematical performance, and therefore is necessarily preceding it. We will use an analogy to make our point: Most members of PME engage in a similar activity in the period of end December – beginning January: the writing of a Research Report (RR) to submit for the forthcoming conference. While some experience a pleasure in writing up their RR as such (this pleasure-oriented behaviour is intrinsically motivated), many see the writing of the RR at least also as a means towards a further end (this productivity-oriented behaviour is extrinsic motivated): The RR has to be written in order to communicate one’s research results to the research community, and/or an RR (or any other contribution) needs to be written in order to be allowed to attend the forthcoming conference as such. The motivation that is experienced in these cases by necessity is always preceding the actual PME conference. If in the course of December, the announcement would be made that the forthcoming PME conference is cancelled, the (extrinsic) motivation to write a contribution will almost certainly disappear. In a similar way, motivation for mathematics is necessarily preceding a mathematical performance, in the sense that it originates with that forthcoming mathematical performance (be it in the near or in the far future) in mind.

STATISTICAL EVIDENCE

We have just shown how a conceptual analysis of the notion of motivation already shows how motivation necessarily has to precede mathematical performance. A next question could be whether such claims would also be supported by empirical evidence.

A lot of criticism can be given on research that relates attitudes towards mathematics to mathematical performance, particularly from a methodological point of view (for an overview, see Zan et al., 2006), and we will come back to this in the next section. Still, it seems worthwhile to look at the general trend in this empirical research.

An older meta-analysis by Ma and Kishor (1997) looks exactly at the issue under consideration here: the relationship between attitude toward mathematics and achievement in mathematics. Specific questions asked were what the strength of this relationship is (in correlational terms), what *causal* evidence there is, and what the magnitude of the causal relationship is. A total of 113 primary studies were included in the meta-analysis. Regarding the overall strength of the relationship, the conclusion based on 108 effect sizes was that the relation was significantly positive and reliable, but it was not a strong one. More importantly, the meta-analysis also investigated specifically the causal relationship between attitudes and performance. Among the 113 studies, only 5 studies applied a causal modelling of the data; all others looked merely in a correlational way. The 5 studies that applied causal modelling reported 10 effect sizes derived from testing 20 227 students. The finding was that the causal relation in the direction achievement→attitudes was not significantly different from zero, while the causal relation in the direction attitudes→achievement was statistically significant, with an effect size of 0.08. Even though, as the authors commented, the magnitude of this effect size was small and therefore cannot be described as practically meaningful, we still think this is a very important finding from a theoretical point of view. It provides clear evidence for the point we make in our paper that affective variables causally precede high performance and not vice versa. We explicitly want to contest the strong focus on (standardized or other) effect sizes and the practical conclusions that can be drawn from them. Silberzahn et al. (2017) have clearly shown this by involving 29 research teams working on the same data set to answer the same research question. Each team came up with its own analysis strategy. The effect sizes that the teams obtained from empirical studies varied greatly. The conclusion they made was that the effect sizes highly depend on subjective analytical choices, so one can argue that "the" effect size does not exist. Still, the vast majority of teams arrived at the same conclusion on the existence and direction of an effect. So, while effect size claims can be discussed, a *theoretical* claim about the existence of an effect can be made reliably.

So while Ma and Kishor (1997) found a small effect, the fact that it was significant and in the causal direction that we expected is a very important finding for our central claim. In this respect, we also want to refer to Mook's (1983) argument that psychological investigations are too often criticized for lacking direct practical relevance: Often, such psychological studies are intended to test specific predictions that derive from a theory. The theory is assumed to be true for various kinds of settings, including laboratory and real-world settings. The prediction, however, is tested in a controlled lab setting. This does not imply that the instruments, manipulations, etc. of that lab study would *directly* generalize to the real-world setting. Most often, the study was not intended as such at all. With Mook (1983, p. 379), we therefore wish to warn for "A misplaced preoccupation with external validity (…) to dismiss good research for which generalization to real life is not intended nor meaningful".

THE PROBLEM WITH SELF-REPORTS

In the previous section, we have explained how there is a significant statistical relation between motivation and high performance, and that it is specifically in the causal direction that motivation precedes performance. We further explained that the small effect size generally obtained is not necessarily problematic. However, there are also other criticisms that can be made about the statistical evidence that we have provided above. A lot of research on motivation (and attitudes in general) is based on questionnaires. While a lot of questionnaires nowadays may have good psychometric qualities, one can always pose questions about the validity of such questionnaires when it comes to measuring motivation or attitudes. The central problem is that such questionnaires are based on self-reports by the learners, and such self-reports can be questioned on various levels. First (and maybe ironically), there may be motivation issues: Questionnaire data are only valid if the participants wish to put effort in reporting how they really feel and think about a certain problem. Data are no longer valid if (some) participants are not motivated to take part in the study, and give random responses. Second, there may be desirability issues: Participants may take into consideration what they think the researcher wants to find, and adapt their answers to comply with this expectation. Third, questionnaires assume that participants are aware of their affect/motivation and are able to report about it. Murphy and Alexander (2000, p. 8) note that in motivation research nowadays "one assumption seemingly (…) is that individual's motives, needs, or goals are explicit knowledge that can be reflected upon and communicated to others". We agree with Hannula (2006, p. 166) that "The present view emphasises the importance of the unconscious in human mind. Motivation, like much of our mind, is only partially accessible to introspection." Fourth, questionnaires are retrospective. One can argue that participants in questionnaire research may not be able to report reliably and validly about their affect when learning mathematics or when solving a mathematical problem while responding to a questionnaire, simply because they are not experiencing this at the moment of reporting about it.

Research nowadays often moves away from quantitative approaches, and turns towards more qualitative approaches, for instance by means of interviews, diaries, etcetera. However, just like the quantitative questionnaire-based approaches, such qualitative approaches rely on self-reports and therefore bring with them the very same problems that were just discussed.

EXPERIMENTAL EVIDENCE AS AN ALTERNATIVE

Regardless of whether one uses a quantitative or qualitative approach to map learners' affect, there are various methodological problems regarding validity of the data. Another substantial concern with a lot of research on the relation between motivation and performance is that it is correlational in nature (see e.g. the meta-analysis by Ma &

Kishnor, 1997). This makes it difficult to deduce a causal influence, and to exclude that the effects are explained partially or in full by a third variable.

In order to address all these issues, it may be desirable to seek experimental evidence. In the limited space that is available to us in this paper, we restrict ourselves to examples. One major problem when pupils solve word problems is that they tend to exclude real-world considerations from the solution process, and tend to give unrealistic answers (see e.g., Greer, Verschaffel, Van Dooren, & Mukhopadyay, 2009). Some studies have experimentally manipulated the setting in which pupils solved such problems, in order to stimulate them to include more realistic considerations, and as such come to a better mathematical performance. For example, DeFranco and Curcio (1997) have offered the following word problem in a typical school test format: *328 Senior citizens are going on a trip. A bus can seat 40 people. How many buses are needed so that all the senior citizens can go on a trip?* Nearly all students answered in an unrealistic way, for instance "8.2 buses". The day after, the same pupils received a real-life problem that – from a mathematical point of view – was completely parallel: The pupils received a facts sheet containing information on a party that needed to be organised for a group of classmates in a specific restaurant. Pupils had to make a phone call to order minivans to transport all children to that restaurant. In this case, nearly all children ordered a whole number of buses.

Such a study shows clearly that the pupils had all relevant knowledge to come to a good mathematical performance, but in the typical school setting they were not inclined to engage in making realistic considerations; they felt it sufficient to just report the results of arithmetical operations. However, when the same students were involved in a more authentic setting, they engaged not only in doing the correct arithmetical operations, but also considered whether that outcome was realistic, and adapted their answer accordingly. In that sense, a realistic mathematical problem embedded in an authentic setting should motivate students realistically at first, and then their mathematical performance and understanding can be really good.

This is a simple study that illustrates how one can experimentally manipulate the motivation of students in relation to the mathematical problem they are solving, and show the causal impact on the quality of the solution they obtain. In this case, the causal impact of a specific kind of motivation on mathematical performance has been shown. Several other studies have been conducted that show how making mathematical tasks more authentic has an impact on students' performance (see, e.g., Palm, 2002). If one wants convincing evidence for the opposite relation (of students' performance on motivation) too, experimental studies should experimentally manipulate mathematical performance, and measure the motivation that follows from it. To the best of our knowledge, such evidence is not (yet) available.

CONCLUSIONS AND DISCUSSION

Our arguments have shown how positive affect or motivation necessarily precedes high mathematical performance. First, even though motivation is complex, conceptually there is a very clear link between motivation and a subsequent high performance. It is believed that motivation provides the primary impetus to initiate learning and later the driving force to sustain the learning. Simple logic further makes us conclude motivation must precede (and cause) high performance. Secondly, from the statistical evidence, even from a meta-analysis, the causal direction that motivation precedes performance is clearly shown. One can argue about the effect size but we have explained why this is not really an issue. The crucial point is that the theoretically assumed relation has been empirically verified. Thirdly, we propose that, to investigate the relation between motivation and performance, sole reliance on self-report measures entails a danger, and experimental evidence seems to be the way to go, while the experimental evidence for the opposite direction (high performance leading to motivation) seems totally lacking. When individuals are doing mathematics, the affective system is not merely auxiliary to cognition, it is central (Goldin, 2002). Thus, more direct experimental evidence would be desirable to show how students' motivation or other affective factors influence achievement.

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POSITIVE AFFECTIVE VARIABLES REGARDING MATHEMATICS *FOLLOW* HIGH PERFORMANCE

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Partly due to the tradition of experimental approaches to (mathematics) educational research, leading to a period of time in which variables were often assumed to be associated by causal-effect relationships, it has been thought by some that positive affect needs to come before any mathematics achievement. Developments in research designs, such as the person-oriented approach, have however suggested that the relationship between affect and achievement might not be so. In this paper, we draw on academic literature to argue that some constructs theoretically support our position. We will present our arguments for the fact that positive affective variables follow high performance (in mathematics), and against the fact that positive affective variables precede performance. We will also assert that a chronological relationship exists between high mathematical performance and positive affective variables.

INTRODUCTION

In most if not all mathematics education systems around the world today, the pursuit of high or higher performance remains a priority. Even for economies like Singapore which have consistently been ranked at the top of international assessment exercises such as PISA and TIMSS, changes continue to be introduced to the local mathematics education systems to ensure that their students continue to improve their achievements. At the same time, the mathematics education community had also moved on from an apparent emphasis historically on the cognitive aspect of learning and teaching, to one which is complemented by academic attention to the affective factors. An indication of this can be seen in the increasing proportion of published reports in the PME proceedings which incorporates affective variables such as attitudes, motivation and self efficacy (see also the recent overview by Liljedahl & Hannula, 2016).

There have been suggestions that nurturing positive affect in students is necessary if they are to achieve in school mathematics. Yet, for example, East Asian students generally do not enjoy positive affective experiences in their studies, while the top performing economies in both TIMSS and PISA are all East Asian (e.g. Hong Kong, Japan, Korea, Singapore, and Taiwan). Indeed, Korean students ranked last in the 'being happy at school' survey (OECD, 2013). At the same time, many of us would have had the experience of doing well at something, and the feeling of a sense of accomplishment and of achievement then provided us with an affective boost. Thus, we

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are of the position that positive affective variables regarding mathematics follow high performance.

We will elaborate on and explain our position in the rest of this paper in the following way. First, we will provide a short review of key literature in the area of the affective domain, showing that some constructs theoretically support our position. Next, we will explain how we arrived at our position stated above. This will be organised in two parts. Firstly, there are arguments for the fact that the development of positive affective variables follows high performance (in mathematics). Secondly, there are the arguments against the fact that positive affective variables precede performance. Moreover, we will refer to studies which claim that problem solving and affective variables cannot be studied in terms of cause and effect.

THE AFFECTIVE DOMAIN

The issue of affective variables in mathematical performance has been widely addressed by research. Most of them, however, focused on difficulty in solving problems, or proposed the idea of an intertwining between affect and cognition. A key reference is the work of De Bellis and Goldin (2006), who asserted that emotions interact with cognition. They speak of ‘affective pathways’, that is, sequences of states of feelings that interact with cognitive representational configurations. In the same strand we pay attention to the works of Furinghetti and Morselli (2007, 2009). We propose here that the affective pathway, in cases of success, ends with a positive state or feeling.

In addition to cognition and affect, the learning process is also associated with a third aspect, which might be named psychomotor (e.g. Harrow, 1972) or conation (e.g. Hilgard, 1980). However, it appears to be more often the case that the context for the learning process is considered as a dichotomy shared between cognition and affect. In some cases here, learning appears to be conceived in terms of reasoning and feeling alone, without any reference to motivation or acting. In other cases, some conative variables (e.g. motivation in Hannula, 2012) seem to have been brought into the ‘affect’ umbrella. What all these mean is that it can be difficult to define what is meant by ‘the affective domain’, and that any reading of this domain should take this into account.

Hannula’s (2012) development of a multidimensional categorisation of the affective domain constitutes a good start to our discussion here. His conception sees affect as being made up of three categories, and the affective domain, three dimensions. According to Hannula (2012), affect may be categorised broadly into motivations, emotions, and beliefs. These form the first of three dimensions of the affective domain. The second dimension highlights ways of considering affective variables in terms of trait versus state. Middleton, Jansen, and Goldin (2016) referred to this feature of affective variables as ‘long term’ and ‘in the moment’ respectively. The third dimension is more philosophical in nature, emphasising the physiological (embodied), psychological (individual), and social aspects of the affective domain. Our point is that there is a chronological relationship between high mathematical performance and positive affective variables, both in terms of trait and state: at the end of a high mathematical performance, there are positive affective variables.

Considering affect as a state, we rely on the construct of attitude, developed by Di Martino and Zan (2011, 2014) in terms of a three-dimensional model, that encompasses emotions related to mathematics, view of mathematics, and perceived competence in mathematics. Such a theoretical characterization "takes into account students' viewpoints about their own experiences with mathematics, i.e., a definition of attitude closely related to practice" (Di Martino & Zan, 2014, p. 575). The authors devoted a lot of research effort to develop the theoretical construct and mainly used it to address the issue of negative attitude towards mathematics: attitude towards mathematics is to be considered negative when at least one of the three dimensions is negative. The construct of attitude provides theoretical support for our thesis concerning the fact that positive affective variables come after high performance. More specifically, we argue that positive attitude towards mathematics comes after high mathematical performance. The experience of a meaningful mathematical activity, indeed, may convey a better view of mathematics, thus improving one dimension of attitude. Moreover, the construct of attitude suggests, from a theoretical perspective, that a high mathematical performance (that is to say a good problem solving process, accompanied by understanding and final success) could improve perceived competence ("I was able to solve this problem"). Going further, we may even see the possibility of affecting attitude towards mathematics by creating conditions for high mathematical performances (see Di Martino & Mellone, 2005 for a related experiment).

ACHIEVEMENT LEADS TO POSITIVE AFFECT

It is our thesis that the development in students of positive affect needs not precede high mathematics achievement. In starting off her writing, Pinto's (see this volume) quotation of Andrew Wiles highlights the fact that no human activity done well is free from an investment of hardwork. As the saying goes, 'no pain, no gain'. In almost every culture we might consider, we can see how hardwork and perseverance underlie success stories of achievement. Of course, we will not say that these hardwork and perseverance are automatically associated with positive affective experiences. For most, if not all, people, the commitment of time and effort as well as the monotony of practice after practice are hardly enjoyable in-the-moment.

Hagenauer and Hascher (2014) drew on the control-value theory to reason that achievement promotes control beliefs and value cognitions. Also, mastery experiences, which include achievement in assessments, have been known to constitute the most powerful sources of self-efficacy growth in students (Martino et al., 2016).

Another argument comes from Liljedahl's (2013) work on creativity in problem solving, seen as a feature of the process, not only of the final product, and a feature that can be applied to the work of any individual, and not only a "genial" one. Liljedahl (2013) studied the nature of mathematical illumination, intended as one crucial of the invention process, encompassing initiation, incubation, illumination, and verification.

Notably, he addressed his research question investigating the nature of illumination for two different samples: mathematicians and future primary teachers. His position, indeed, is the one of Hadamard: "*Between the work of a student who tries to solve a problem in geometry or algebra and a work of invention, one can say there is only a difference of degree*" (Hadamard 1945, in Liljedahl, 2013, p. 256). Liljedahl (2013) has described the AHA effect, or the illumination phase as derived by Hadamard (1945), that occurs when a mathematical problem is finally solved. As Poincaré (1952) pointed out, the illumination phase is accompanied by a feeling of certainty and positive emotions. Our point is that such positive feelings come during/immediately after the illumination phase. The study of the AHA effect supports our thesis, since the good performance is intended to end with a positive feeling. Liljedahl also categorized issues reported by the prospective teachers when detailing their AHA experience. Such reports concern emotions ("*This was the best feeling*", p. 258), change in beliefs ("*I used to think that math was all about the right answer, but now I am more aware of the value of the process*", p. 259), change in attitudes ("*Also, I enjoy math now. I feel like this success stimulated more success. Now I have raised my expectations in math*", p. 259), mathematical understanding. Concerning mathematicians, their reports, although different from the prospective teachers' ones in terms of content, are also "affectively" connotated. In summary, the illumination experience is an affective one and the affective experiences occur when the problem is finally solved.

POSITIVE AFFECT WHICH APPARENTLY LEADS TO ACHIEVEMENT

Some research appeared to imply that affective variables come first, before achievement in mathematics. For example, Hannula et al. (2014) conducted a 6-year longitudinal study with more than 3,500 Finnish students, in which a reciprocal relationship between mathematics achievement and self-efficacy was observed. However, the dominant effect of this relationship was actually from the former to the latter.

The role of positive interest in predicting subsequent student achievement in mathematics is also not a certain one. Köller, Baumert, and Schnabel's (2001) study, for instance, found that interest and achievement are positively correlated only in a context where the participants involved believed that there is choice. In this study, thus, it was only in the higher levels of schooling that student interest in the subject predicts their subsequent achievement, since at these levels the students had a choice between different mathematics subjects to take up.

The study conducted by Pintrich et al. (1998) might, at first glance, imply that positive affect in the forms of students' goals brings about achievement. However, students' previous grades turned out to be an even stronger predictor. Thus, positive affective states/traits need not come first before we see achievement in mathematics amongst students.

There is also possibly a theoretical reason for the several studies which appeared to show that the positive affective variables come before mathematics achievement. This is based on the observation that historically, some of the conative variables might have been considered as being affective in nature. These may include motivations and values. Seah (2018), amongst others, had considered these as being conative, to the extent that motivations and values drive an individual to activate the relevant 'I want to' mindsets (see Seah, 2018). Such mindsets might include 'I want to achieve', 'I want to do my best' and 'I want to be able to solve all mathematics problems'. Bishop (1988) had emphasised the power of values in developing the will and perseverance in an individual. Thus, to the extent that such conative variables as motivations were considered as affective, there is a possibility that they were responsible for driving the students involved to have the will, motivation and grit to achieve in mathematics, thus creating the impression that in some studies, positive 'affective' variables had led to mathematics achievement.

AFFECT AND ACHIEVEMENT NOT ALWAYS CORRELATED?

Instead of affect and achievement being related by a causal association, which also renders a reciprocal relationship meaningless, there might well be no relationship between these two. A good example for these can be found in the PISA 2012 data. The top 5 economies where their students were happiest – namely, Indonesia, Albania, Peru, Thailand and Colombia – registered a range of performance, from the 50 to the last position amongst 65 participating economies (OECD, 2013). At the same time, there was also no correlation between the top/last economies in the performance ranking and the student happiness ranking (OECD, 2013).

Although they are not as often used by researchers, person-oriented research designs can and have deepened our understanding of learning and teaching in ways which traditionally variable-oriented designs have not been able to. Roeser, Eccles and Freedman-Doan (1999), for example, conducted one such person-oriented study with some 500 students to explore the association between school motivation and emotional functioning on the one hand, and school achievement on the other hand. With this research design, it was found that no association could be established, except for one of the four subgroups which represented students with low school motivation and poor emotional functioning. This research study thus suggests that affect and achievement may not be associated with each other for all student categories, and when they do, it was not for the group of students who possessed high school motivation and also for those who had strong emotional functioning.

We can find resonance of this association existing for unmotivated students of mathematics in more recent studies. For example, Parhiala et al.'s (2017) study with more than 1,600 15-year old students in Finland, could not detect any relationship between subgroups of students classified according to motivation and emotional well-being levels, yet "students with low motivation were more likely to show low levels of math and reading or math performance" (p. 200).

Shifting our focus away from the 'Western' education systems to the 5,197 students in Malaysia who took part in PISA 2012, say, Thien and Ong's (2015) analysis showed that a maintenance of positive affective variables did not guarantee student achievement in mathematics, definitely not for mathematics self concept, as well as instrumental and intrinsic motivation. In neighbouring Singapore, where 5,546 students took part in PISA 2012, instrumental motivation was found to have a negative and significant effect on performance (Thien & Ong, 2015).

Perhaps one of the most compelling demonstration that positive affective variables do not necessarily precede student achievement may be seen in Hattie's (2009) 800 meta studies, an ongoing project which has now analysed nearly 1,200 studies. Amongst the 252 factors contributing to – and impeding – student achievement in general, affective ones can be found across the spread. This suggests that there are affective variables which when developed (e.g. motivation) can lead to mathematics achievement, but there are also affective variables (e.g. anxiety and depression) which when developed do not facilitate significantly to or become barriers to such achievement. In other words, it would be too sweeping a statement to claim that positive affect comes first before we can see high achievement.

In fact, the absence of a causal-effect relationship between affect and achievement has led to a reconceptualisation of some affective constructs. Take a common one, attitudes, for example. There was a time when studies were designed to explore the causal-effect relationship between the attitudes and mathematics performance. At the turn of the century, however,

> the inadequacy of the assumption about cause-effect relationship between attitude and behavior has emerged; attitude is now considered to be an interpretive instrument to understand the reasons for intentional actions: intentional actions involve complex relationships between affective and cognitive aspects. (Di Martino, 2016, p. 3)

A CHICKEN-AND-EGG PROBLEM?

There is also evidence in the literature which suggests that what we are dealing with here is really a chicken-and-egg problem. Wigfield and Cambria (2010) started their writing with this following paragraph:

> Work on social cognitive models of achievement motivation has burgeoned over the last 30 years. These models emphasize individuals' beliefs, achievement values, goals, and interests as major influences on their motivation and ultimately, their achievement (Eccles & Wigfield, 2002; Pintrich, 2003; Weiner, 1992). These models also emphasize the influence of the achievement context on individuals' motivation and achievement (Pintrich, 2003; Wigfield, Eccles, & Rodriguez, 1998; Wigfield, Eccles, Schiefele, Roeser, & Davis-Kean, 2006). (p. 1)

That is, there is a reciprocal relationship between positive affect and mathematics achievement, with each promoting the other in turn. What this seems to imply is that whether we promote positive affect or bring about mathematics achievement in a student, we are effectively starting the cycle towards further growth and development of positive affect and achievement in mathematics, and which is likely able to sustain itself. Yet, despite the many decades of research in – and classroom-level development of – positive affect, there has not been this realisation of cycles of mathematics achievement amongst students across education systems and cultures. Even if the cause of this were to be the complex relationship amongst affective variables or between them and cognitive/social factors, this also surely implies that positive affective variables do not come first before school mathematics achievement.

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RESEARCH FORUMS

OTHERNESS IN MATHEMATICS EDUCATION

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Reflecting on the inseparability of the I from the other, this research forum (RF) proposes to highlight how the connection between them can be discussed in mathematics education. Building on and extending the existing research, we aim at clarifying and reflecting on the role and the place of Otherness in teaching and learning mathematics. Looking at mathematical, social, and cultural tools, historical texts and problems solving as well as broader socio-political and epistemological considerations, our discussions will provide a base to highlight the richness and the nuances in relation to the issues of Otherness in mathematics education.

THE PROBLEM OF THE I AND THE OTHER

This RF aims at discussing the various theoretical perspectives about the interrelationship between the individual and the other(s). It highlights how political, cultural, and social issues such as equity, ethnomathematics, history of mathematical education, identity, ethics, and beliefs could not be fully comprehended, without a thorough understanding of the relationship between the I and the other (Planas & Valero, 2016; Gutiérrez, 2013). There is an extensive body of research concerned with the idea of otherness. Lerman (1996) pointed to the relationship between the individual and the other to propose that research involves "frameworks which build on the notion that the individual's cognition originates in social interactions [with others] … and therefore the role of culture, motives, values, and social and discursive practices are central, not secondary" (p. 4). This proposal was the entry point for theories that investigated the effects of external factors into mathematics education in an attempt to give an account of the "individual in context" (Bartolini Bussi, 1998).

Sociocultural research in mathematics education conceptualizes the relationship between the I and the other predominantly in two ways: relationships with others through which we *learn,* and relationships which produce and effect of *othering*. Concerning the former, research has shown that we learn in relationship with others. Examples include variety of use, modifications and conceptualizations of Vygotsky's (1978) more knowledgeable other in the Zone of Proximal Development (ZPD) (Goos et al., 2002; Graven & Lerman, 2014). But also the work that, inspired by Lave and Wenger (1998), examines meaning making and learning in communities. For example, Jaworski and Goodchild (2006) examin mathematics teachers' professional learning communities and Bose and Subramaniam (2011) studied children knowledge-building

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communities. Concerning the latter, research shows how the relationships for learning do not only fulfill a purpose of changing meaning, understanding, and mathematical competence; but also has an effect on subjectivity when different views of the learners and teachers, their position, and their capacity in relation to mathema-tics are created. These effects of othering are as important to consider as the effects of the other when learning, because the othering creates categories of differentiation with simultaneous exclusion and inclusion. Examples of this research include the work of Planas and Civil (2015) on how bilingual students become othered through the classroom dynamics, resulting in exclusion from access to a full participation in school mathematics. Walshaw and Anthony (2006) discussed the power of discourse and hegemonic discourses of power, and Setati (2006) pointed to the critical role of ideologies in institutions of mathematics education as important in creating possibilities of access for historically underprivileged students.

From a more theoretical perspective, the notion of otherness is also discussed in Radford and Roth (2017): "Thus, the action of an individual (self) is always social because the other is involved or implied, not merely when another person is present (Roth, 2016). An action is for (the purpose of) another, having been initiated by an action of the other, and thereby returning to the other" (p. 372). This important notion is not often taken into account in studies of learning, developing identity, and the interaction between teacher and learner(s). We will argue that through a focus on the notion of the other, possible explanatory frameworks for development or their absence emerge. To bring forward, with all the force, the idea of otherness, we emphasize on the fact that the questions of otherness cannot be fully understood if we think of mathematics as students' individual, subjective constructions.

In this RF, we focus on different theoretical and philosophical perspectives to extend the existing conceptualization of others and othering. Abtahi's (cf. 2014, 2018) proposes the other in the more knowledgeable other to include *tools*. Herheim considers how rules and procedures in mathematics can be *mathematical othering* for students. Guillemette emphasizes on the experience of otherness that is provided by history and historical mathematics texts. Lerman offers a view of others in teachers engaged in professional development. Maheux analyses *altering* to describe the movements by which otherness is *in the making* alterity, the *state of being other* takes place during mathematical activity. Finally, Valero problematizes *epistemological disadvantage* as a category for the other. Our discussions provide a platform for both organisers and the participants to question the notion of otherness, examining various theoretical frameworks, and cross-examining the similarities and differences in various theoretical conceptualisations.

KEY QUESTIONS

The following questions are the main focuses of the forum: Who/what is an other? What is otherness and othering? What are different aspects of otherness? Within concrete examples of mathematics teaching and learning, how can they be perceived? What are potential possibilities, challenges, and consequences related to otherness? What does it mean to do research regarding the question of otherness? How can teachers' and students' actions be seen as positions of othering?

TOOLS AS THE OTHERS

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PURPOSE, FOCUS

The purpose of this part of the research forum is to consider *tools as the other*. (Mathematical) tools are inseparable parts of learning mathematics. I consider mathematical tools as any tool-like object that its mathematical affordances are perceived by someone who is doing or is thinking about mathematics, like a piece of paper or an apple.

From a Marxian/Vygotskian perspective, our actions and being are inherently social in origin. Marx (1844) stated: "Even when I carry out an activity which I can seldom conduct in direct association with other men, I perform a social […] act. It is not only the material of my activity—like the language itself, which the thinker uses—which is given to me as a social product (in interaction with others). My own existence is a social activity" (p. 11). Partly building on Marx's ideology and to concretely formulate a vision of how our actions, the material of our activity and even our being are indirectly mediated and linked by the others; Vygotsky introduced the concept of the mediated action.

Wertsch (1992) argued that Vygotsky's concept of mediational means/actions illustrates how human actions are ultimately tied to the others. That is, to the cultural, social, and institutional settings within which we live and grow. These others provide and shape the cultural tools that are used by individuals to form and acquire particular types of functioning, such as understanding a mathematical concept. In this approach, the mediational means are the elements that carry the other-ness of the social and cultural patterns and the knowing of a society. The mediational means are used, created, and modified over time by others. Hence, their use becomes an indirect interaction with the others. I emphasize the fact that all human actions, including our teaching and learning of mathematics, employ mediational means in essential ways. Given the fact that these mediational means are products of society, culture, history, and politics, then our actions (including our learning) are essentially and indirectly initiated and shaped by the culture, history, and politics of the others.

Within an educational setting, Vygotsky assumes that people's *learning* is also a product of social interactions, with humans and non-humans and with signs and tools. To Vygotsky, learning happens in the presence of others, whom may have better knowing of some cultural and social practices. Radford (2013) viewed learning as a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection. In his notion of the ZPD, Vygotsky's makes it clear that others play a major role in children's learning—that is, we only learn in the presence of others who are more knowledgeable (or not) (Abtahi, 2017). One of the basic tenets of the Vygotskian approach to education is the assumption that individual learning is dependent on the social interaction. However, it should be clear from the outset that this is not merely a statement of correlation between individual learning and social context [with others]; this thesis should be interpreted in its strongest possible form, proposing that qualities of thinking are actually generated by the organizational features of the social interaction.

I believe it is worth considering the possibility that the organisational features of our social interactions with mathematics in its learning and its teaching could be located not only in human-human interactions, but also in human-tool interactions. A tool can also become the other that could be more knowledgeable. My rationale for claiming tools as others refers to the knowing(s) that is accumulated in the socio-culturally created and/or used tools; within which reside, the traces of the knowing and perception of the people who have socially and over time designed, used, and modified the tools. For example, at times, children use interrelationship between the sizes of the Cuisenaire rods to do mathematics. The rods carry within their design the mathematical thoughts of Mrs. Cuisenaire. So children's interactions with the tools become cultural interactions between one and the other. The following example is an excerpt from a transcript that shows how S and F dealt with the tension of creating a decimal number using the Cuisenaire rods. After some trial and error, S thought to consider the rod of 2-unit to represent 1, and consequently for the rod of 1-unit to represent 0.5.

[showing the white rod of 1-unit]	S: if this is one
[showing the red rod white rod of 2-unit]	S: if we make this one
[showing the white rod of 1-unit]	S: this would be… a point five F: a point five… yah

This interaction shows how S and F perceived the specifics of the design of the rods to think about mathematics. In this interaction, the properties of the design of the rods guided the children to solve their problem, hence with a ZPD, I assume the rods to be the other that is more knowledgeable. I reiterate that the more knowledgeable-ness of the rods is the accumulated thoughts of Mrs. Cuisenaire. In this interaction, I assume a mediated network of social relationship between mathematics, the child, the task, and the tool, in which all the elements are the others to one another. And their other-ness is socially and historically led and plays a mediating role in children's thinking about mathematics or doing mathematics.

MATHEMATICAL OTHERING – MEMORIZING RULES AND PROCEDURES IN SCHOOL MATHEMATICS

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PURPOSE, FOCUS, AND THE CONCEPT OF MATHEMATICAL OTHERING

The purpose of this part of the research forum is to elaborate on how certain ways of doing and talking about school mathematics can result in *mathematical othering*. Othering is regarded as a discourse in which certain wordings, choice of words, and practices dominate the communication. Pandey (2004) described othering as a manner in which dichotomies can be generated and represented through language, often in unintended ways, via binary oppositions as 'them' and 'us'. This approach can be adapted to describe a potential division between mathematics and students, between 'it' and 'us'. Mathematical othering concerns the relationship between students and mathematics, and represents a distance between students and mathematics. But what does mathematical othering look like? Moreover, what can create such othering?

The concept of othering offers an approach to understanding reproduction of patterns and relations in which mathematical othering takes place. When mathematical othering is emerging or is already established, the mathematics is not part of the students or the students are not part of the mathematics. Students can be othered from mathematics by the way it is taught and communicated, or mathematics can be othered from the students by the students themselves because they do not engage or identify themselves with mathematics.

Masingila's (2002, p. 37) found that a majority of the students in her study "think mathematics is synonymous with school mathematics", and Wager (2012, p. 10) built on this by saying that students often do not see "the connection between the mathematics in which they engage in school and the mathematics they experience out-of-school". Mathematical othering can therefore take place at two different levels. At an overarching level, mathematics is only a part of students' life at school and not an integrated part of their lives in general. Mathematical othering also occurs and accumulates within the mathematical classroom. The latter is the focus of attention in this text.

A critical linguistic approach, inspired by Nilsen, Fylkesnes, and Mausethaugen (2017), is applied in order to gain an understanding of potential mathematical othering between students and school mathematics. A particular focus is directed towards the use of rules and remembering procedures when students work with the four arithmetical operations. Students' choice of methods and their choice of words when explaining their choice of method are the units of analysis.

The use of rules and procedures with an emphasis on memorizing and speed is deeply rooted in the culture and history of mathematics education and still plays a dominant role (Boaler, 2010; Herheim, 2016). Mellin-Olsen (1991) identified this prominent way of thinking about mathematics teaching when he developed the concept of the exercise paradigm. He observed how teachers talked about mathematics teaching by using certain metaphors, often referring to travelling: "having a delay", "speeding up", "being way ahead", and "catching up". The focus was to cover the curriculum and do as many tasks as possible, leading to students memorizing rules and procedures.

MATHEMATICAL OTHERING AND I-IT RELATIONSHIPS

I convoke ideas of Bakhtin (e.g. 1986) and Gadamer (2004) to investigate how pupils can be mathematically othered by the use of rigid rules and the remembering of procedures in school mathematics. Combining Bakhtin and Gadamer might seem problematic, since Bakhtin is considered to be a dialogical thinker while Gadamer is strongly influenced by dialectics. However, some of Gadamer's ideologies are regarded as dialogic based on his emphasis on conversation and how he holds many of the common views shared by most dialogical thinkers (Vessey, 2005). I therefore argue that combining the thinking of Bakhtin and Gadamer provides a powerful framework for investigating school mathematics and othering.

A core of mathematical othering is the (lack of) relationship between the students and the mathematics. Bakhtin (cf. Friedman, 2001) and Gadamer (cf. Stewart, 1985) both relate to Buber (2004) on how one can meet others and the fundamental differences between I-It and I-You relationships. These relationships can be transferred and adapted in order to discuss the relationship between students and a subject, and the I-It relationship is of particular interest when investigating mathematical othering. Buber described I-It relationships as reduced engagement and distance (termed remoteness by Buber) between the I and the It. According to Gadamer, I-It relationships involves key concepts like *inauthentic dialogue* and *apparent questions*. According to Bakhtin, I-It relationships can be described and understood by concepts like *the authoritative word* and *isolated utterances*. Applying these concepts makes it possible to analyze and discuss a potential lack of authenticity and genuine questions and the presence of authoritative words and isolated and detached reasoning in school mathematics. They give grounds for talking about mathematical othering as a potential consequence of memorizing rules and procedures.

EXAMPLES ON MEMORIZING RULES AND PROCEDURES

When working with the four arithmetical operations: addition, subtraction, multiplication, and division, there are numerous mnemonic rules students can encounter. Typical examples are: “Remember to move the comma one to the left when you divide by 10”, “Remember to add a zero when you multiply by 10”, “Remember to borrow”, “Remember to make a mark on the number you borrow from”, “Remember the carry digits”, “Remember where to write the carry digits” and so forth. These instructions all have in common that they are expressed as commands, as authoritative orders about detailed and isolated steps that have to be obeyed. By emphasizing details like adding commas and zeros, the rules encourage students to focus on how to use the rules, not to understand why commas are moved or zeros added. The choice of words “remember to” indicates the high importance of the rules and that is something one *has* to do. The wording also indicates that the rules need to be remembered rather than understood. The presence of such rules can therefore enforce an I-it relationship between students and mathematics. Rote remembering and use of rules can lead to isolated chunks of knowledge that creates a distance between the students and the mathematics.

The next example concerns subtraction. A grade six student did the following three calculations (Melbye, 2001):

$$\begin{array}{r} {}^{10}\ \\ 8\cancel{5}7 \\ -834 \\ \hline 017 \end{array} \qquad \begin{array}{r} {}^{1\,10}\ \\ 5\cancel{9}3 \\ -485 \\ \hline 112 \end{array} \qquad \begin{array}{r} {}^{1\,1}\quad \\ {}^{\cancel{1}\cancel{0}\,10} \\ \cancel{4}06 \\ -239 \\ \hline 233 \end{array}$$

The student does the same on all the three tasks—he applies a systematic approach. In the first task, he seems to borrow although it is not necessary. How he gets the seven ones in the answer to the first task is unclear. Following standard procedures, you get 10 + 7 − 4 = 13 or 10 – 4 + 7 = 13, but clearly, that is not what he does. In the second task, a plausible guess could be that he turns things around and takes 5 − 3 in order to get a positive answer. However, in this task, regrouping has to be done, and it seems like that is what he does. Counting the ones should give 10 + 3 – 5 = 8, but again, it does not correspond with what the student has done.

Situations like this call for a talk with the student to understand how he thinks. The student’s explanation was clear: “When I deal with subtraction tasks, I have to borrow” (Melbye, 2001, p. 75, my translation). The student says, “I have to borrow”. His recollection is that when you do subtraction, the rule says you have to borrow, no matter what numbers are involved. The first step of the procedure is (always) to start borrowing. The student explains that in the first task, he does 10 – 7 = 3 and 3 + 4 = 7. That explains the seven ones in the answer to the first task. Similarly, in the second task, he does 10 – 3 + 5 = 12, writing the two ones in the answer and the ten as a carry digit. This serves as an original combination of detailed mnemonic rules from addition and subtraction in the same task. The same approach is applied in the third task.

The student tries his best to use the rules the way he remembers them and make them work—and he is good at it. He remembers all the steps: borrowing, making a mark on the number he borrows from, subtracting the ones, carry digits, subtracting the tens, and so forth. However, the student's rule implies that he does not have to consider the numbers involved when deciding what to do—one always has to borrow. The rule is authoritative, there is only one correct way of doing it. It is what Gadamer would call inauthentic dialogue and Bakhtin would call monologic. There is a lack of genuine, critical questions like "Why do we have to start with the ones?", "Is this an appropriate method?", and "What can be clever to do when these numbers are involved?"

Remembering procedures and algorithms by memorizing seemingly unrelated steps can be regarded as isolated utterances from a Bakthinian perspective. The student is not trying to get in touch with what is happening, aiming to understand what a step is about and why it is done. If the way mathematics is communicated is dominated by indisputable and rigid rules, details that have to be remembered and steps that have to be done, then students that emphasize remembering rules and making rules work in order to produce answers is a natural consequence. Each step is its own isolated utterance, creating a distance between the student and the mathematics, resulting in mathematical othering.

CONCLUDING COMMENTS

Mathematical othering can be characterized by doing mathematics without reflection. The students use rules without asking critical questions, without discussing if the rule or the method is good and adequate, or if it is inappropriate or needs tweaking before being used. They quit thinking and just do mathematics—they become doers without reflection. Dewey's (e.g. 1933) famous quote "learning by doing and reflection" turns into "learning by doing without reflection". There is little or no reasoning about what characterizes the task and the numbers involved, and based on that, what could be the easiest or most effective method to use. Note that, rules and procedures can, when combined with a focus on understanding, become tools that are more knowledgeable others (cf. Abtahi's argumentation in this research forum).

The lack of reflection constitutes the primary source for the distance between students and mathematics—for mathematical othering. Consequently, mathematical othering can generate a misconceived belief of understanding while rote learning is what really takes place. Applying the authoritative rule, following the procedure step by step, is erroneously regarded as understanding—because the mathematics has become so remote and distanced for the students that they are not able to tell the difference between rote learning and understanding. Mathematics is regarded as something you just have to believe in, like a religion. No one can explain why it happens.

THE ETHICAL SUBJECT OF LEVINAS

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The notion of Otherness is taken, as Levinas (1971/2010; 1978/2011a; 1983/2011b) put it in his multiple phenomenological essays, as the central and the core of human being existence. The French philosopher overturned the traditional, from Plato to Heidegger, ontological way of thinking human being. With Levinas, the philosophical inquiry on human being doesn't begin and capitalize on his nature (ontological perspective) but on his relation to the Other (ethical perspective). In other words, ethics here is not taken in as a "satellite" element of human existing, it is rather the central and the determinant field of reflections. The complex and profound philosophical reflections from Levinas can provide basic elements and concepts in order to understand teaching and learning mathematics (*e.g.* Atweh & Brady, 2009; Roth & Radford, 2011; Ernest, 2012).

Levinas's philosophy provides, above all, a subject perceived primly and fundamentally in his ethical aspect, in its relation to the Other (peoples, objects, concepts, ideas, history, sciences… everything that have a meaning). Ethic is not understood here as the philosophical principles that could or should aim at building moral code or behavioral conduct imperatives. It is rather a path (in a phenomenological sense) to understand human being and particularly here the subjectivity. That is to say what constituted the human being experiences, its possibility, its limits, and the way human being built himself.

In his latest work (1978/2011a), in response to the famous critique of his early works by Jacques Derrida in *Violence et métaphysique* (1979), Levinas develop a notion of the subject literally constituted of the Others. He then quite his traditional phenomenological investigations on Otherness, based on the tradition inherited by his masters Husserl and Heidegger, to open a very new perspective on human being which his no longer perceived as an isolated subject being beset by phenomena, an ipseity thrown in the reality, as would have said Heidegger, but an ethical subject constituted, and revealing himself, in a deep sense, in the relation to the Other.

Levinas would say that subjectivity is a hostage. It means that we are the host of the Other. What will have a mean, our very private horizon of mean, is made possible and constituted by the Other. Each of our movements of consciousness is then understood as an engagement.

Sociocultural theorists in mathematics education, such as Radford and Roth (2011), have built on Levinassien perspective in order to develop articulated elements of a framework that could avoid representationalism and rationalism drawing a private, self-regulated and autonomous (in a rationalism and dualism perspective) subject

(Radford, 2008; 2012; 2013). This position is marked by strong emphasizes on a sensible and historical aspect of the mind leading to a particular perspective on the subject.

AN ETHICAL SUBJECT

On a more down-to-earth way, I will try to draw on different perspectives that carries a Levinasien concerning subjectivity on the subject who is learning mathematics.

One can find, through "classic" perspectives, a subject that is perceived as "already given". That is to say, a determined subject that is historically and culturally in search of an understanding of himself. In this perspective, the subject, properly educated and intimately aware of its position in the world, will be able to deploy a certain freedom, making him responsible for his actions, improving his relation to the discipline and engaging him with lucid, rich and open activity in mathematics. A freedom that is made of independence, critical thinking, openness and curiosity, characteristics of any good (liberal) scientist. It is this recognition of ourselves that could lead to the opening to the outside, opening to others as opportunity for mutual growth (typically liberal, rational, atomist, perspective on the subject). This is why we can talk about a subject that is "already there", a determined subject (culturally, historically, linguistically, socially, etc.) that is waiting to be discovered, hidden, inaccessible or potentially not yet noticed. This perception of the subject is the core of the most import paradigms in mathematics education. Radford (2008) explains how cognitivist and constructivist models are based on such perceptions.

However, this perception of a subject "already there" can be opposed to a perception of a subject which, in some way, "is not already there". With Levinas for instance, the subject is not considered as completed, determined, given himself to himself. On the contrary, the subject is conceived as "a becoming", constituted and constituting itself in a deep and fundamental ontological sense. In view of this, the mathematics classroom can be seen as a place where it is possible to overcome the particularity of our own understanding of mathematics, understanding limited to our own personal experiences and sociocultural context in which we live.

The focus here is not necessarily and exclusively on the relation to ourselves, but, more broadly, on a relation with the Other in mathematics, a movement toward the community. Thus, the attention is not on an individual experiencing personal possibility of emancipation, but to the possibility for learners to discover new ways of being-in-mathematics, to open, with the others, the space of possibilities in mathematics and to respect the classroom as a politics and ethics entity, open to novelty and subversive issues. Thus, learning mathematics, in such perspective, is not simply learning "to do" mathematics (even less to solve mathematical problems), but rather "to be-in-mathematics", the mathematical activity being nothing else than a way "to be-with-others".

Indeed, the sociality of the learning process means the formation and transformation of consciousness, which is precisely (con)sciousness, that is to say "common knowledge" or "to-know-with-others". In this context, the mathematics classroom doesn't assign itself the role of promoting an individualistic idea of autonomy, but rather one as social engagement (*cf.* Arendt). These elements become for me, in a Levinassien perspective, central elements of mathematics education…

HISTORY OF MATHEMATICS AND MATHEMATICS EDUCATION

I will now present how this Levinassien perspective helps me to understand phenomena related to teaching and learning mathematics. In this study (Guillemette, 2017), I was looking to describe the lived experience of preservice teachers engaged in the reading of historical texts. The next excerpt from video analysis show how Katia's and Mitia's mathematical activity, interactions and interpretations concerning Fermat's *minima* and *maxima* method (the section studied concern Fermat's general method, introduction presented in his text *Méthode pour la recherche du minimum et du maximum*). A dialogue is emerging between Katia, Mitia and Fermat:

> Mitia says he understands his approach, but does not understand why it leads to the solution. Katia says she's not convinced either.

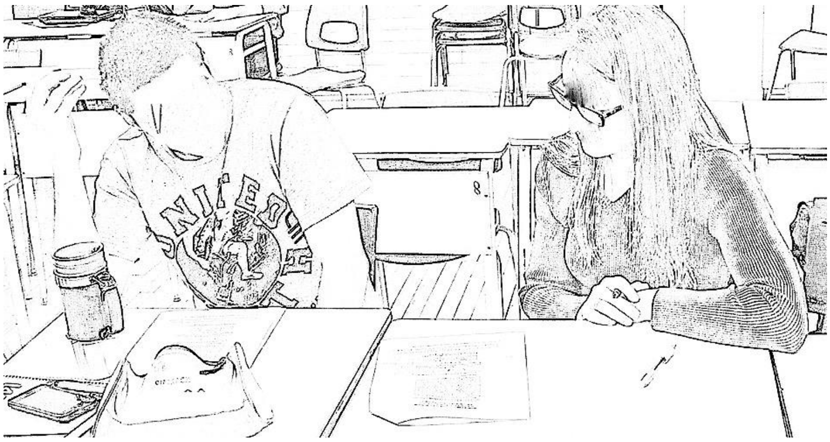

Figure 1: Katia and Mitia reading Fermat

> Mitia then read aloud the paragraph where Fermat explains his method in general and seeks to know the meaning of the unknown e.
>
> Katia gives the hypothesis that e is a variable and that one must find for which value of e the area of the rectangle is maximum. Mitia does not see what it comes about. Katia then tries to represent the rectangle whose area is to be maximized and a second one whose side is increased by a given length of e.
>
> She emphasizes that it is difficult to represent the division by e from its geometrical representation. Mitia notices that e is 0. Martha, from the other side of the classroom, indicates that Fermat previously divides by e. Mitia wonders, "How is it that it divides by 0?". He concludes that "e is not worth 0, but not far". Katia continually tries to illustrate the procedure geometrically, while Mitia invalidates her reasoning, claiming that the value of e is zero. Katia disagrees.

Mitia and Kitia highlight the paradox concerning infinitesimal quantities manipulations. With Fermat's method, which participates of the beginnings of calculus formalization, they are confronted with an exploratory reasoning showing genuine and foreign way of dealing with these objects. The encounter with this fragmented and emerging mathematical discourse brings, phenomenologically, an impression of distance to the participants. They cannot do noting else but to convoke their own modern modalities of expression (especially here representation of algebraic quantities) in order to enter in a dialogue with Fermat's utterances, responding themselves to other utterances (reference to Diophantus's notion of *adaequalitas* for instance). This distance, which is here a temporal one, as well as the polyphonic aspect of the text itself, emphasis for the participants how individual activities, mediated by the sociocultural context, constitute the genetic root of the mathematical activity, containing rational, aesthetic and functional expressive dimensions.

This is where history of mathematics seems to bring the most for reflection on mathematics and mathematics education. Indeed, this impression of distance associated with the reading of historical texts, which I understand as an experience of Otherness, appears when students are confronted with different modalities of being-in-mathematics, which go far beyond what could be brought out by unfamiliar modern mathematical ideas for instance. This temporal distance highlights unfamiliar profound expressive dimensions, rather than modern unfamiliar reasoning could have done. This perception of an ethical subject based on Levinas help me to understand the reading of historical texts, in the context of teachers' education, as an experience of "radical Otherness" for teachers candidates.

Going further with this point, with Radford (2012), I think that we can think about learning mathematics as learning to-be-with-the-others-in-mathematics. By the Other, I mean, of course, the others in the classroom, but more broadly the Other as the community in its ideological struggle, the community engage with the cultural form that mathematics consist.

ALTERING (IN) MATHEMATICS EDUCATION

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DIFFERANCE

Jacques Derrida's work invites us to look at the words we use, question the world they bring forth, and all the gaps, the ambiguities and the paradoxes that come with that. Thinking about otherness, and the way we (begin to) look at in mathematics education is an interesting challenge. Derrida's take on the idea of otherness or alterity is unique in discussing the dynamics, changing, differential being and becoming of otherness in ways similar to how the moving nature of identity is conceived by Ricoeur (1998), for example. Derrida thus talks about "differance", a made-up word meaning at the same time to *differ* and to *defer* (Derrida, 1982). Differance is about externalities in both time and space. In actual time and space, otherness (alterity, differance) is not "the state of being other or different": it is never something static, not even for a split-second. Differance is always in the making. Differance is encountered as an imposition, as finding, as something brought about: an experience whose particular quality is to exceed us. Alterity comes and goes beyond us, before and after, it lives through us as we live it. How can we make sense of such conceptualisation in mathematics education? How can we learn from it?

DOING|MATHEMATICS

Enmeshing such philosophical reflections with mathematics and mathematics education requires an epistemological discussion. In the recent years, I have begun to outline a theoretical perspective on mathematical activity in which the emphasis is set on how (so-called) teachers and students "do mathematics" together, as opposed to teach and learn it (e.g. Maheux & Proulx, 2018). This work strongly hinges on a reflection about self and others through which it appeared that focusing on knowledge (for example: what students know, or should know) raises profound ethical questions (e.g. Maheux, 2010). An answer to this situation is to rather pay attention to how people *do* mathematics, and how they do *mathematics*, an approach we designate by the dialectical expression "doing|mathematics" (e.g. Maheux & Proulx, 2014). This way of thinking takes us to look at mathematical activity in terms of movement and traces (e.g. Roth & Maheux, 2015). Examining this in the light of differance helps us conceptualize how doing|mathematics is realized through teachers' and students' transactions with one another, but also with the material world, what I previously called "being-with-in" (mathematically) (Maheux & Roth, 2011). I would to like focus here on the later part.

OTHERNESS-IN-THE-MAKING: A FRAGMENT

In the next fragment, we observe two second-grade students (Nelly and Kelly) and a research assistant (Mary) working with a tangram. A prompt written on the blackboard invites them to try and make various shapes (a square, a hexagon, etc.). Derrida's ideas take us to look in this "mundane" episode for instances of *the making of* otherness in time and space, otherness-in-the-making, and appreciate how these instances correspond with what we see as mathematical movement. Here, I simply illustrate how teachers' and students' actions can be seen in terms of "making differance" in a mathematical way.

Nelly: I made a very very interesting shape! [fig 1a]

Nelly goes back to the pieces and gently moves them, but they do not come together in a neatly fitting way.

Mary: Or maybe here, you know… and… no…

Joining Nelly, Mary reaches to the pieces, also moving them gently, and then steps back.

Nelly pushes away some of the triangles and starts adjusting the parallelogram and the medium triangle, and then adds a small one (the 'missing piece') [fig. 1b]

Nelly: Here I'm using the missing piece [fig. 1b], they fit together! And then [*adding another small triangle*]

Mary: A boat! [Nelly keeps going, adds another triangle [fig. 1c], and then briefly pauses] Ooh! Interesting!

Nelly: Let's count the sides. 1,2,3,4,5,6 [*pointing to the vertices*]. An oc…

Kelly and Mary: Hexagon

Nelly: Hexagon.

Kelly: A true hexagon.

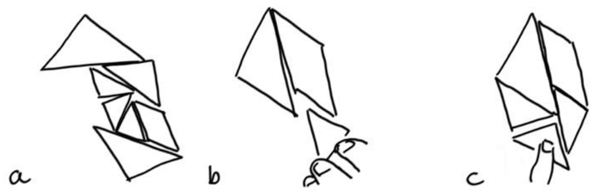

Figure 1: Nelly's very very interesting shape (a), to which she then adds the missing piece (b), and another triangle (c), thereby producing a "true hexagon"

HOW ACTIONS MAKE DIFFERANCE IN A MATHEMATICAL WAY

At the beginning of the fragment, Nelly points out "a very very interesting shape" she made with an excited voice. There is something in the disposition of the tangram pieces that makes the configuration somehow familiar to her: it is a "shape", but one for which she does not provide a specific name. The shape clearly needs some work to fully reveal itself (and Nelly act on that), but its potentiality to becoming something (some thing) in itself is laid out already, and as such offered for action. Nelly's recognition of a strange entity, and the work of bringing it about in which Mary joins her, are ways of making mathematical difference while attending to the pieces. Nelly's utterance creates a temporal otherness by distinguishing the appearance, the revelation, of a shape (a "very very interesting" one): the birth of something other (than a messy, disorganized cluster of objects) which is worth attention, even though she does not exactly know what it is. This lack of knowledge, so to speak, is precisely what makes the distinction (a configuration that *differs* from previous ones) a displacement in time (a *deferring*) which create room for further mathematical work: the search for this strange, slightly familiar "shape" (that turns out not to be). Nelly and Mary's gentle manipulation of the pieces are about transforming what is given into something new, something that is almost there, one could say. The quality of exceeding us at the heart of otherness is clearly visible here: the configuration imposed on the pieces is in return imposed on to Nelly and Mary who respond to it in a way that is recognizable to us as mathematically relevant. The strange arrangement exceeds them even though they created it (by naming it a shape, and attending to it): the configuration found them searching for recognizable mathematical shapes, at the same time that they found it. And in this encounter, we can see how otherness really is “in the making”.

The otherness-in-the-making of the “shape” step by step, moment by moment, results from and generate mathematical actions. This becomes even more visible (by contrast) in the second part of the transcript. When Nelly pushes away some of the pieces and starts working on another configuration, the “interesting shape” completely disappear from her mathematical work. Traces of her previous attempt are still sitting on her table, but their quality as potentially relevant “other” to work on/with mathematically seems to vanish. The excess meaning (to use Bakhtin’s expression) of the configuration does not concern Nelly anymore (although she could eventually go back to it, if the traces remain), it is no longer part of her visible experience. No longer object/subject of her actions (or Mary’s for that matter), their (mathematical) otherness blend into the sameness of a meaningless background. And we see through this how actions make differance in a mathematical way.

What takes place in the last part of the transcript, when Nelly “count[s] the sides” and starts naming the new shape (“an oc…”), another aspect of how differing and deferring relate to mathematical activity is nicely illustrated. First, we see again how mathematics is something we *do* through mathematical actions, like reciting number names in order, pointing to specific parts of an object, combining adequately both actions to count something, using appropriately a specific mathematical term, and so on. A series of actions where each of is, in Bateson’s words, “a difference which makes a difference, and it is able to make a difference because the neural pathways along which it travels and is continually transformed are themselves provided with energy” (Bateson, 1972, p.465). An important deferring from Mary’s comment in turn 4 when she calls then configuration “a boat” before marking a pause and then saying in a very perceptively higher voice “Ooh! Interesting!” as the new shape emerges. The third part of the fragment is about dealing with a much more familiar shape, one for which the students have a name, a name that can be related to the number of sides or vertices they can count. All this, however, still clearly needs to be brought forth for these children, and that need is again what opens to the occurrence of the mathematical work that takes place. Even familiar objects have a potential of otherness, of being more than what we “know” from them in the first moment of an encounter.

Of course, there is a significant difference between meeting someone for the first time and asking them “who are you?”, and walking into an old friend and genuinely ask “how are you today?”. But in both cases, otherness-in-the-making is fundamental to the experience of meeting someone. The dimensions (time, space, ideas…) in which we move are core to the experience the *encounter* where otherness appears in the form of an excess through which we navigate, should it be in friendship, in hatred, or in mathematics. But otherness-in-the-making is also the progressive displacement of otherness in time and space. This is what we see taking place as the girls engage with the shape, associating themselves with it by checking out some of its features, calling it names, etc. The configuration is step by step connected with a nexus of mathematical ideas with which it "belongs", and it seems like otherness is slowly being un-made and the students "get to know" the shape, express familiarity.

Much more could be said about this fragment. For example, we could look into how Nelly baptises the configuration with a name of its own (a name that is in a way impossible because at the same time unique to it and universal): "a true hexagon". Or discuss how otherness-in-the-making of course not only concerns the objects that become part of students' mathematical activity, but also people. And more generally, it would have been important to show how making differance set out the terms by which *mathematics* is what we see taking place. But we'll have to stop here for now.

ALTERING (IN) MATHEMATICS (EDUCATION)

I offer the term "altering" to describe the movements by which differance, or alterity-in-the-making, concretely takes place when people do mathematics. The term particularly speaks to me because of its many resonances. Altering is of course "making other", and it speaks not only to htow doing mathematics implies memorizing transforming he world around us. Altering is also what the socio-material world does to us as we do mathematics, the transformation of how we think or act in the world (both individually and disciplinary, but I won't discuss this here). Altering is thus a good candidate to replace, in this view, words like teaching and learning. A change that is not simply about dressing an old king in new clothes. A new language is a new way of thinking, of doing, of being in the world and with one another (Rorty, 1989). Through altering, what we are usually concerned with in terms of learning (and teaching) mathematics is reformulated as "becoming acquainted" by "associating oneself" with mathematical ideas and ways of doing. Doing something (new or not) changes us, it inescapably affects us by "reinforcing" how we deal with the world, and/or "opening up" new possibilities. Mathematics education can then be seen as a place where people simply meet to do mathematics together, some people being more acquainted with it than others. A place where 'learning' mathematics is not about facts, acquiring skills or even understanding ideas, but more like becoming to 'know' a person, and being a friend .(I borrow these from images from Seymour Papert, who used them on several occasions.)

A STORY OF BECOMING "OTHER" – OF LEARNING MATHEMATICS TEACHING WITH AND FROM OTHERS

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CASE STUDY – FOUR TEACHERS

In this paper, our specific focus will be on teachers engaged in PD based on developing their knowledge of mathematics knowledge and pedagogic knowledge, within the work of the Wits Mathematics Connect Secondary project (e.g. Adler, 2015). The story we tell emerges from a "learning study" (Runesson, 2014) conducted with four teachers teaching mathematics in two secondary schools located in one of the poorest urban townships in the Gauteng Province in South Africa. We draw from this wider study (Pillay & Adler, 2015) as it provides an interesting and illustrative case of teachers taking up (or not) different mathematics teaching practices. Following the account we will offer a possible theoretical framework in terms of otherness.

Wits Maths Connect Secondary (WMCS) is research-linked professional development project aimed at improving mathematics teaching in disadvantaged secondary schools in one province in South Africa. WMCS is informed by a sociocultural orientation to learning as mediated, and of mathematics as a network of scientific, connected and hierarchic concepts (Vygotsky, 1978).

A year-long 16-day *mathematics for teaching* course that takes place away from the school has become the cornerstone of the WMCS intervention, offering participating teachers an opportunity to strengthen their own relationship with mathematics, and with mathematically coherent teaching. The impact on teachers' teaching and student learning has been reported elsewhere (Adler, 2016; Pournara et al., 2015). Alongside this work, we have explored lesson study as a complementary professional development context for teachers who have been in the course. Here we have worked with small groups of teachers from a cluster of schools, on the possibilities for learning mathematics and mathematics teaching through participation in focused study of co-constructed lessons in a cycle of planning, teaching, reflection, replanning, reteaching as is common in various lesson study projects. We have described how this work is informed by models of Japanese lesson study (e.g. Fernandez, 2002) and Swedish learning study (Runesson, 2008; 2013) and we have reported researchers' and teachers' co-learning as well as the tensions and dilemmas that accompany this work (Alshwaikh & Adler, 2016; 2017).

The specific story is interesting precisely because while all four participated in the same learning study cycle of planning, teaching, reflecting and reteaching four different Grade 10 classes a lesson on functions, their participation varied, illustrating how within one community of practice, taking on/up an-other's teaching process and within it an approach to mathematics and its learning varied in substantive ways. Otherness as offered in the learning study (informed as it was by the larger PD project—WMCS) with respect to mathematics teaching was embraced by two, partially attempted by one and ignored by a fourth teacher. We hope through their story of participation in the learning study to illuminate the notion of otherness.

The learning study focused on the teaching of functions in Grade 10. In the South African national curriculum, the linear, quadratic, hyperbolic and exponential functions are studied with focus on the parent function and then a vertical shift (e.g. the quadratic functions explored would have the general form:

$y = ax^2\ ,\ or\ y = ax^2 + q, a \neq 0).$

In the mid-year common assessment that all Grade 10 learners wrote in both schools, items related to functions were performed very poorly, with clear evidence to the learning study group (the four teachers, and a researcher from the WMCS project) that learners had difficulty recognizing different representations of the same function. Moreover, for the question requiring a sketch of the graph of the equation $y = 2^x$, some learners drew a straight line, others the parabola $y = x^2$, notwithstanding having constructed a correct table of values for the exponential function.

The learning study group decided that they wanted to plan and then teach a lesson where learners' attention was focused on the form of the equation for different functions and their related graphs. In particular, following variation theorists (e.g. Marton et al.) and Watson and Mason's (2006) focus on discerning key features of mathematics objects through variance amidst invariance, ideas to which the teachers had been introduced in the wider PD and elaborated by the researcher at the start of the learning study, the group decided to focus on the following five equations and their graphs—where while x and 2 were invariant, the expressions formed in their combination were different:

$y = 2x\ ; y = x^2\ ; y = \frac{2}{x}\ ; y = \frac{x}{2}\ ; and\ y = 2^x$

The lesson plan developed in the first joint planning session of the group was to present an equation (starting with $y = 2x$, then its graph and then a second equation $y = x^2$ and its graph, focusing on what was the same and different in the graph as the equation changed and so on bringing into focus the algebraic form of the function and its graph and the deeper understanding that as the relationship between the variables changes, the function changes and thus the graphs differ. The juxtaposition in the sequence was seen to be important.

Mr A volunteered to teach the first iteration of the lesson to his Grade 10 class. He followed the plan in terms of sequencing the functions to be drawn, but the whole lesson was taken up with completing a table of values for each of the functions, and so no related graphs were drawn. In the reflective discussion and planning for the second iteration, Ms B planned a lesson for her Grade 10 class where she began with two equations $y = x^2\ ; y = \frac{2}{x}$ and their graphs, bringing attention to their differences. The reflective discussion following the lesson identified that while working on the different representations at the same time was what they had discussed, learners were not following, nor were they able to respond appropriately to the questions posed about the equations and their graphs. Discussing what might be the root of learner difficulty, the group, following a suggestion from the researcher, agreed that learners were not sufficiently fluent with the form of the different equations, and so were not able to discern the difference between the equations themselves. The plan for the third cycle was that there needed to be more deliberate attention to the equation form as well as its graphic representation, starting with the function $y = 2x$ and then juxtaposing this with $y = x^2$, and followed by the remaining three, and completing hopefully at least two of these during the lesson. Ms C taught this lesson to her Grade 10 class, but as the lesson unfolded it was clear she had not prepared what had been jointly planned and taught a similar lesson to Mr A with a focus on completing a table of values and so substitution into some of the equations. In the reflective discussion Ms C shared that she had not had enough time to plan. It became clear through the study cycles that Ms C, in addition to her full-time teaching position, ran a small business that took up much of her out of school working time. Mr D taught the fourth iteration, paying careful attention to each equation form and its graph in relation to the previous equation presented and discussed, and thus following the joint plan. The whole study group was satisfied that Mr D's Grade 10 class had been offered the most opportunity to learn what they had intended in the learning study.

All the four Grade 10 classes were tested at the end of the cycle of lessons, where the task was to match different equations to their respective graphs. Mr D's class performed significantly better on this test as expected. A surprise, however, was the Ms B's class also improved substantially. Mr A and Ms C's classes were far less successful with the task. In their discussion of the results, and pondering why Ms B's class had done relatively well, she shared how, following Mr D's lesson, she had taught this same lesson to her class as she had felt it had been so successful.

ANALYSIS

It is generally the case that research reports on successful take-up of PD. Here we wish to discuss what occurs in other cases, when teachers appear not to follow their colleagues in their learning. The orientation of the project, as pointed out above, is sociocultural. For us all learning, both for school students and adults, here teachers, is through mediation in zones of proximal development. ZPDs emerge, or not, in response to a whole range of social factors (Lerman, 2001). These include individual interests and concerns, goals and motivation, focus or distraction, and what has been described as catching each other's attention (Meira & Lerman, 2009). In PD we can describe ZPDs of groups of teachers and, by the nature of the teaching activity and the goals and motivations that occur in the practice of teaching, individual teachers. What matters most, of course, given the need for learners to acquire mathematical knowledge, is that the PD catches their attention in sufficiently strong ways for them to see the goals of the PD aligning with their own when it comes to their practice. For us, 'otherness' describes those whose goals do not.

In the story here, we see take-up from Ms B and Mr D, in interestingly different ways—learning from the study and its orientation and importantly from each other's teaching practices. Ms C without any deliberate planning taught as was customary and similar to Mr A. Neither took the initiative of Ms B to 'reteach' their class following Mr D's lesson. Further research, both with students and teachers, is required to provide insights into the non-emergence of a ZPD, the absence of alignment of goals, seen through the least opportunity for students to learn.

CHALLENGING THE ASSUMPTION OF EPISTEMIC INEQUALITY IN THE "OTHER"

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While different theories address the notion of "the other" in ways that can be of significance for mathematics teaching and learning, I enter in the field of studies on the cultural politics of mathematics education (Valero, 2017). These studies explore the cultural significance of mathematics and mathematics education in the forming of contemporary societies, from a perspective that highlights how mathematics as a privileged form of knowledge in modernity is connected to the creation of subjectivities. From such perspective the question concerning "the other" in mathematics education interrogates current taken-for-granted truths and forms of reasoning about "the other" and taps into their historical configuration in the epistemologies and practices of mathematics education. Such analytical move has the intention of offering new ways of imagining mathematics education.

I will contend that the persistence of deficit perspectives on students' low mathematical achievement is rooted in culturally entrenched assumptions about the *epistemic disadvantage* of "the other". Rationality and scientific knowledge became the core of notions of progress and development that defined the "Enlighted" superior human in Western culture. Mathematics being at the heart of the historical forming of modern science became the norm of human reason and advancement. The valuing of mathematical/scientific forms of thinking as superior took place in a series of material and political conditions that ordered and differentiated peoples and cultures in nations and inter-nation spaces, deeming different groups, their life forms and knowledge as naturally incapable of acquiring "proper", developed thinking. This process is not simply the context or the pre-condition of the organization of school mathematics curriculum; it is the core of its historical articulation. In what follows I present the main elements of this argument and point to some directions for further studies to unpack how particular mathematics education practices continue operating on the grounds of the assumption of epistemic disadvantage.

THE CURRENT DIFFERENTIATION OF THE OTHER

A problem addressed in socio-political studies of mathematics education is the differentiated access to and success in mathematics of different types of students (Planas & Valero, 2016). A sustained fact of mathematics education is that many students do not achieve expected levels of attainment or simply fail. When scrutinizing who recurrently fails, it becomes evident that the various types of students "at the bottom" have something in common, namely to be considered different, unprivileged, inferior or, at least, at the margins of the dominant cultural norms and values of a society. The interesting observation is that being at the bottom of a table of performance in mathematics —independently of how such performance is measured— coincides with belonging to groups of people with characteristics that differ from such valued "norm". They are "the other".

A quite frequent explanation for such coincidence has been the attribution of deficits to "the other". Research has explained failure in mathematics for women — and more recently boys—, immigrants, ethnic minorities, particular racial groups, students with disabilities, etc. by determining what they miss in order to be able to lift themselves to reach the desired established levels. Even if research adopting deficit perspectives has been criticized for almost 20 years, the identification of deficits as explanation for differential performance still persists, reifying the idea that "the other" at large is not capable of acquiring mathematics at a normal level. Of course, research findings have nuanced deficit explanations in an attempt not to focus only on individual lacks. For example, the discussion of differentiated school results using data from TIMMS and PISA has been seen in terms of "equity gaps", which complement a focus on "achievement gaps" —directing attention to the results in performance—, with the identification of other gaps such as the "opportunity gap" —highlighting the impact of a differentiated availability of resources— and the "learning gap" —pointing to a differential access and participation in quality instructional environments (Lindblad, Petersson, & Popkewitz, 2015). Also, there is a recognition that categories of differentiation have changed in time and space. For example, differentiations on school mathematics performance on the grounds of gender used to be more visible some decades ago but have tended to equalize or even to turn around —the disadvantage of girls has lessened and nowadays there is the tendency of boys to underperform in certain levels and countries. Differences connected to students' socio-economic status are persistent; while new categories of differentiation such as language and ethnicity are expanded in contexts where migration has increased.

Despite these nuances, what remains is the unquestioned logic that positions "the other" in disadvantage and inferiority, in need of direction towards the desired characteristics of the "I", the representative of the norm to be attained. The relationship between the "I" and the "other", when seen within a historical grid of power, is not simply a starting point for the unfolding of society and of human thinking and the materially produced world. This relationship is at the core of the functioning of power in making subjectivities and in justifying a differentiated appropriation and distribution of material and symbolic resources. In the configuration of modernity, the "other" has been conceived as having an epistemic disadvantage, that is, as subordinate in knowledge and forms of life with insufficient capacity for knowing in relation to the "I".

MATHESIS IN NARRATIVES OF CULTURAL SUPERIORITY

The idea of mathematics being a special and superior activity that epitomizes the highest and purest form of thinking and intelligence has been repeated, with some variations, as part of narratives of the Enlightenment. Science has been a unique knowledge for humans to tame and transform the world, and generate progress. Since the 16th century, different forms of mathematical work entangled not only with the areas of knowledge of Natural Philosophy, but also with the generation of new rational, knowledge-based technologies of government of the population. The displacement of "the letters" in favour of the "new scientific forms of knowledge" made of the latter the centre of the cultural, political and economic spirit of a Modern world (Gaukroger, 2006). This does not mean that religion or the humanities disappeared; it meant that in the project of the European Enlightenment, sciences and mathematics acquired a central position in the way in which the Western culture came to understand itself as well as came to establish a relationship with other peoples and cultures.

As part of a cultural rationality, the developments of mathematics as accounted in the traditional history of mathematics are not as important as how such developments both embed and promote cultural ideas. For example, one can talk about the emergence and configuration of mathematical analysis and calculus as the sequential process of formation of the concepts, notation and procedures discovered or invented by (mainly) men in different European sites; or one can try to understand these as part of the practices of a time in relation to the formation of a larger form of reasoning, what Foucault called an episteme. *Mathesis*, "a universal science of measurement and order" (Foucault, 1994, p. 56), brings together mechanical understandings of the word with a desire to mathematize, in such a way that "relations between beings are indeed to be conceived in the form of order and measurement, but with this fundamental imbalance, that it is always possible to reduce problems of measurement to problems of order. So that the relation of all knowledge to the mathesis is posited as the possibility of establishing an ordered succession between things, even non-measurable ones." (p. 57)

Mathesis unfolded in forms of thinking and knowing that classify qualities, objects and people, measurable or non-measurable, and organize them in a succession. Furthermore, the quantification of qualities generated a sense of objectivity, factuality, truth and universalism in the claims of knowledge (Poovey, 1998). This logic was present in mathematical notions such as Leibniz's continuity in calculus based on his natural philosophy of "nature not making leaps" (Markley, 1999). It was also present in the expansion of mathematical analysis, as the "art of discovering truths characterised by the systematic of the decomposition of any idea to be considered, and by the power of abstraction of formulas" (Brian, 1998). Analysis became the basis of procedures for the calculus of probabilities that were part of the administration of the French population around the time of the Revolution. Mathesis articulated in the emergence of areas of knowledge and their political use in the regulation and administration of human life, both in mathematics itself, in natural philosophy and in the emerging social and human sciences.

My point here is that mathesis is a central characteristic of the classificatory, ordering and comparative logic that characterizes modern, western forms of knowing and being; and that such logic is not just a way of thinking in the minds of people, but it is instantiated in their material and discursive practices; and that it has been closely connected to practices of government and power. In this sense, mathesis is never neutral; it is at the centre of the operation of power.

MATHESIS AND "THE OTHER"

It is also important to point that such logic operated within the forming nation states as much as it operated across other populations that Europeans met in their cultural expansion through colonization. Within nation states the rational administration of the population found justifications in different forms of science to justify differential distribution of and access to resources. Take for example the construction of gender differences as a justification for the superiority of men. Chronaki (2008) evidences how articulations of biology, genetics and anthropometrics articulated at the end of the 19th century generated categories of gender and race where people in the category "woman" and "black" were assigned primitive characteristics such as narrow, child-like and delicate skulls, belonging only to inferior castes, in contrast to the more robust and round heads of fully developed males of superior race (p.17). Such classification establishes a norm for measurement —the fully developed male— and orders comparatively "others" in relation to that norm: females and blacks. The effect of the classification is the creation of differently valued positions and the attribution of inferior cognitive characteristics to the other. Chronaki argues that such type of reasoning is still present in the ways in which children and teachers think of themselves as naturally capable of (not) learning and (not) relating to mathematics, science and technology.

Across different populations and nations, the comparative logic of the West was important in maintaining colonization with its human and material exploitation. The narrative of superiority of the European colonizing peoples were central in governing the new population and installing the desire of civilization and the promise of progress, though the classification and ordering that anthropology established between the non-civilized, savages and the universal measurement of civilization, the White man (Fabian, 2014). The main issue at stake is the assumed inferiority of other forms of life and knowledge, embedded in the very historical constitution of the study of "the other"; in other words, the assumption that "the other" is epistemically disadvantaged is at the core of the study of "the other".

The wide network of mathematics education practices functions on a historically and cultural constituted grid of power. A call for contemporary mathematics education research is to unpack how forms of reason such as that responsible for the maintenance of the epistemic disadvantage of the other in different instances of instruction in classrooms and schools, and in society at large.

PRIOR READINGS

Prior readings (papers, resumes or excerpts) will be available using the PME infrastructure when the final program will be available. The readings will include all excerpts for the readings activities as well as all different material that will be used.

CONTRIBUTION TO PROCEEDINGS FORMAT

All researchers contributing will be invited to extend their reflections on Otherness building on philosophical or theoretical positions that they developed above (4 pages including references for each researcher). Secondly, an online discussion between all researchers will take place in order to draw on the main questions that could be addressed to the community around Otherness in mathematics education. A resume of this discussion and a development on these main questions will be presented (8 pages including references), as well as a short introduction and a description of the format of the proposed Research Forum (2 pages).

PROPOSED FORMAT

Day 1	• Presentation of the extensive overview of existing research on the theme. (15 min.) • Q & A (5 min.) • Short presentations on theoretical perspectives relating to Otherness including examples of mathematical activity to identify instances of "other-ness" and "othering" (20 min.) • Small group followed by whole group discussion (40 min.) • Q & A and recapitulation (15 min.)
Day 2	• Presentation on operationalization of these theoretical positions in research - pedagogical objectives that are formulated in reflections on Otherness (15 min.) • Small group analysis of data from empirical studies with students or teachers' candidate—data will be provided in order to stimulate *in vivo* the reflection of the group (40 min.) • Whole group discussion, Q & A, closing remarks (25 min.)

To enrich the group discussions, we post data for participants to have access to ahead of time. Points arising from both sessions will be fed back to our concluding remarks, in which we combine the discussion from the two sessions to highlight possible further research and actions.

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LEARNING AND TEACHING OF ARITHMETIC SKILLS IN EARLY YEARS

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This Research Forum aims to consolidate the current research on early arithmetic learning and teaching. Five research groups will participate in discussions focusing both on theoretical approaches and innovative research methods that have been developed and practiced during the recent years. The RF will contribute an overview of earlier and on-going research to highlight contemporary academic debates and perspectives within this field and present advances in research. Furthermore, we direct our attention to the future and need for knowledge and methodological development.

INTRODUCTION

PME aims to open opportunities for international connections and exchange of research in mathematics education. Our Research Forum is a contribution to this PME spirit by directing attention to the youngest learners in mathematics education. Knowledge of both the development of arithmetic skills and also the teaching of emerging skills in the early years is essential for mathematics education in general, which is at the heart of the PME interest. Thus, we find there is a need to highlight early mathematics learning and how new methodologies may advance the field, which still to a large extent rely on research based in paradigms that peaked in the 1980's and 1990's (c.f. Carpenter, Moser & Romberg, 1982; Fuson, 1988).

There is, also, a burgeoning body of research suggesting that early arithmetic experiences matter for later development and academic performance (Duncan et al., 2007). We know, for example, that children at a very early age discriminate quantities, and even unions of quantities, with increasing accuracy during their earliest years and that numerical development is influenced by social interaction in a long-term perspective (Pruden, Levine & Huttenlocher, 2011). However, it is not obvious how these early abilities develop into arithmetical skills. Given this gap, how to develop and teach arithmetic skills remains a pedagogical and empirical question.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 125-154). Umeå, Sweden: PME.

In this RF we gather researchers that represent the state of the art, not only in researching children's arithmetic learning, but also the teaching of arithmetical skills. With this group of researchers, we aim to highlight theoretical and methodological approaches that help us understand how encountering a social-culturally influenced education in the early years promotes children's development.

OUR CONTRIBUTION TO THE MATHEMATICS EDUCATION COMMUNITY

Different counting strategies are found in empirical studies of children's arithmetic problem solving and these are thoroughly described in the literature, such as 'counting all' and 'counting on' or recognizing the inverse relation between addition and subtraction (see Baroody, 2016; Carpenter, Moser & Romberg, 1982; Clements & Sarama, 2009 for overviews). The field of research agrees upon promoting children's development of the conceptual idea of numbers, for instance, that children's understanding of structural relationships within calculations allows them to develop more advanced ideas about addition and subtraction (Davydov, 1982; Schmittau, 2004; Baroody, 2016). Although research provides a generally solid consensus regarding developmental trajectories, our knowledge of how children develop accurate and general understanding of basic arithmetic principles and make use of them as strategies for arithmetic problem solving remains incomplete (Baroody, Torbeyns, & Verschaffel, 2009; van den Heuvel-Panhuizen & Treffers, 2009).

Research within this field most often takes a psychological, cognitive or neuro- scientific perspective. A pedagogical perspective is, however, also necessary, because it pinpoints how to facilitate learning and development. It is not enough to know what skills children should acquire or how different skills are interrelated (Duncan et al., 2007; Rusconi et al., 2005); there is an urgent need for heightened attention on how to teach, so as to support children's learning and development of these concepts and skills. Many programmes have been conducted during the last decade to learn more about arithmetical development and to test and improve mathematics teaching practices (c.f. Clements & Sarama, 2008 "Building Blocks"; Venkat & Askew, 2012 "Wits Maths Connect Primary Project"; Tsamir et al. 2014 "CAMTE" and Hannula-Sormunen, Lehtinen & Räsänen, 2015 "SFON"; among others).

The goal of this RF is to create dialogue and synthesis about early arithmetic learning and teaching. Five research groups share findings from research projects and innovative methodological approaches to researching young children's arithmetic learning. This area is of global interest with inquiries expressed in both large-scale and smaller qualitative studies. We will highlight contemporary perspectives and summon current epistemological and methodological questions within this area. The two key issues we will address are:

- *Theoretical approaches to children's conceptual understanding of numbers and arithmetic*. Different theoretical approaches to the teaching and learning of arithmetic skills are highlighted in terms of what knowledge different theories provide, and how different theoretical concepts may enlighten our understanding of children's arithmetic development.

- *Innovative methodological approaches to studying early arithmetic teaching and learning*. Research with young children is a delicate matter and much has been learnt since Piaget's clinical interviews. Contemporary designs for researching children's arithmetic reasoning consider to a larger extent how context, culture and social interaction influence children's responses, with implications for how children's learning is interpreted by researchers and teachers.

The Research Forum constitutes internationally well-known researchers, who as a collective represent the state of the art and, in particular, have brought innovative methodologies and developed theoretical frameworks into the study of arithmetic learning and teaching in early childhood education. In the following, the contributions give an overview of earlier and on-going research, highlighting contemporary academic debates and perspectives within our field and present advances in research.

A LEARNING TRAJECTORIES APPROACH TO EARLY ARITHMETIC

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At the core of our theory of hierarchic interactionalism lies the construct of learning trajectories (LTs), each consisting of three components: mathematical goal, developmental progression (from cognitive science and mathematics education), and instructtional tasks and strategies correlated with each level. Our LTs for early arithmetic including one for subitizing and one for counting-based arithmetic—with competencies for the two mutually supportive. Two decades of research have both validated the LTs themselves and their usefulness, from efficacy in smaller studies to effectiveness at large scale, including three cluster randomized trials resulting in moderate to strong effect sizes. Results are also positive for sustainability and to a small extent, persistence of effects.

THEORETICAL FRAMEWORK: HIERARCHIC INTERACTIONALISM

We synthesized nativist and interactionalist theories to create a theoretical framework that we call hierarchic interactionalism (Clements & Sarama, 2014a; Sarama & Clements, 2009). The term indicates the influence and interaction of global and local (domain specific) cognitive levels and the interactions of innate competencies, internal resources, and experience (e.g., cultural tools and teaching). Mathematical ideas are represented intuitively, then with language, then cognitively, with the last indicating that the child possesses an understanding of the topic and can access and operate on those understandings.

There are 10 tenets of hierarchic interactionalism, but central ones show the criticality of learning trajectories (LTs), each consisting of three components: mathematical goal, developmental progression (from cognitive science and mathematics education), and correlated instructional tasks and strategies correlated with each level. These research-based LTs are hypothesized to be useful tools educators can use to improve mathematics learning and teaching.

Five tenets are most directly relevant to LTs (for a full explication, see Sarama & Clements, 2009).

Developmental progression and *domain specific progression.* Most content knowledge is acquired along developmental progressions of levels of thinking, each of which is most propitiously characterized within a specific mathematical domain or topic.

Consistency of developmental progressions and instruction. Instruction based on learning consistent with natural developmental progressions is more effective, efficient, and generative for the child than learning that does not follow these paths.

Learning trajectories. A particularly fecund approach is based on hypothetical learning trajectories complete with instructional tasks that include external objects and actions that mirror the hypothesized mathematical activity of children as closely as possible. These tasks are sequenced, with each corresponding to a level of the developmental progressions. Specific LTs are the main bridge that connects the "grand theory" of hierarchic interactionalism to particular theories and educational practice. We believe the power and uniqueness of the learning trajectories construct stems from the inextricable interconnection among the goal (the mathematics), developmental progressions of thinking, and correlated instructional activities (Clements & Sarama, 2014b).

Instantiation of hypothetical learning trajectories. Hypothetical learning trajectories must be interpreted by teachers and are only realized through the social interaction of teachers and children around instructional tasks. Societally-determined values, goals, and cultures are substantive components of any curriculum; research cannot ignore or fully determine these components a priori.

METHODS TO CREATE RESEARCH-BASED LEARNING TRAJECTORIES

To determine the first component of LTs, the goal, we relied on both the expertise of mathematicians and research on students' thinking about and learning of mathematics (Clements, Sarama, & DiBiase, 2004; Fuson, 2004; National Mathematics Advisory Panel, 2008; Sarama & Clements, 2009). This results in goals that are organized into the "big" ideas of mathematics: overarching clusters and concepts and skills that are mathematically central and coherent, consistent with students' (often intuitive) thinking, and generative of future learning.

Once the mathematical goals are established, we review research for empirically-based models of children’s thinking and learning in the targeted subject-matter domain. For example, young children can invent their own solutions to simple arithmetic problems (Baroody, 1987; Carpenter & Moser, 1984; Ginsburg, 1977 Kamii, 1985) and profit from doing so more than from being taught prescriptive procedures (Hiebert et al., 1997; Kamii & Dominick, 1998; Steffe, 1983, 1994).

We then review all available research to determine if there is a natural developmental progression (at least for a given age range of students in a particular culture) identified in theoretically- and empirically-grounded models of children's thinking, learning, and development (Carpenter & Moser, 1984; Griffin & Case, 1997). That is, we build a cognitive model of students' learning that is sufficiently explicit to describe the processes involved in the construction of the mathematical goal across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power, and generality.

Next, we use both research and the wisdom of expert practice to determine the nature and content of activities. For example, curricula have been crafted that pose problems in the forms of activities and games and ask children to figure out how to solve the problems and explain their solution strategies (Baroody & Coslick, 1998; Fuson, Carroll, & Drueck, 2000; Hiebert, 1999; Kamii & Housman, 1999), often using scaffolding techniques to guide their inventions (van den Brink, 1991). The set of proposed activities is sequenced according to this nascent developmental progression (Clements & Sarama, 2004). Once that general model has been accepted or developed, it is tested and extended with teaching experiments, which may initially present limited tasks and adult interaction to individual children with the goal of checking the models of children’s thinking and *learning* (Steffe, Thompson, & Glasersfeld, 2000). This is an intensely iterative phase (Clements, 2007). That is, in practice, such models are usually applied and revised (or, not infrequently, created anew) dynamically, simultaneously with the development of instructional tasks, using grounded theory methods, clinical interviews, teaching experiments, and design experiments.

A general guideline across these research and development methods is that equity issues be considered. For example, considerable thought be given to the students who are envisioned as users and who participate in field tests; a convenience sample is often inappropriate. Systemic sociocultural issues should be considered as well.

COMPLEMENTARY LEARNING TRAJECTORIES FOR EARLY ARITHMETIC

Our LTs for early arithmetic including one for subitizing (ANS and perceptual to conceptual or arithmetical subitizing) and one for counting-based arithmetic—with competencies for the two mutually supportive.

The major developmental progression for subitizing is not only movement from small to larger numbers, but the qualitative shift from *perceptual subitizing* (Clements, 1999), recognizing a number without consciously using other mental or mathematical processes and naming it, to *conceptual subitizing*, applying the perceptual subitizing processes repeatedly and quickly uniting those numbers. That is, recognition in which the person uses such partitioning strategies and is aware of the number in the whole and the parts. Some instructional activities are straightforward, such as *Snapshots,* in which a teacher shows 6 objects for two seconds or less and children name it as "six" and justify (e.g., "I saw 3 and 3 here"; "I saw those 4 and then 2 at the top"). (For a full exposition, see Clements & Sarama, 2007/2013, 2014a; Clements, Sarama, & MacDonald, 2017, or LearningTrajectories.org).

Counting-based arithmetic. There are two major influences on levels of thinking, the level of counting and subitizing competence and the type of problem. Children develop increasingly sophisticated counting strategies to solve increasingly difficult problem types. For example, most initially use a counting-all procedure and then curtail them, in favor of other methods such as "counting on," which often involve subitizing as well as counting. For example, to solve 5 + 3, children might count out 5, subitizing it visually, then use counting and rhythmic subitizing to count on, "five…*six, seven, eight!"* (For a full exposition, see Clements & Sarama, 2007/2013, 2013; 2014a, or LearningTrajectories.org).

EVALUATING THE EFFICIACY OF THE LEARNING TRAJECTORIES APPROACH

Evaluations have shown that Building Blocks can be effective, with moderate to large effect sizes (Clements & Sarama, 2007), even when compared to another research-based curriculum not built upon LTs (Clements & Sarama, 2008). Analyses by topic within these studies showed that among the largest gains were conceptual (arithmetical) subitizing and (counting-based) adding and subtracting.

LT-based approaches must also be evaluated as to their contribution to scale up. We created a scale-up model TRIAD, for Technology-enhanced, Research-based, Instruction, Assessment, and professional Development, which places learning trajectories at the core of the teacher/child/curriculum triad to ensure that curriculum,

materials, instructional strategies, and assessments are aligned. When implemented with fidelity, TRIAD has shown moderate to strong effects (Clements, Sarama, Layzer, Unlu, Germeroth, et al., 2017; Clements, Sarama, Layzer, Unlu, Wolfe, et al., 2017; Clements, Sarama, Wolfe, & Spitler, 2013; Sarama, Clements, Starkey, Klein, & Wakeley, 2008).

Further, we must look at longitudinal effects, which are particularly important for large-scale interventions, because a full concept of scale requires not only consequential implementation, but also endurance over long periods of time and a transfer of responsibility from any external organization to the internal resources of a school district. TRIAD's effects did not persistent at the same level; they decreased over time, but effects were better maintained in TRIAD's "follow through" condition that gave some support to Kindergarten and first grade teachers (Clements et al., 2013; Sarama, Clements, Wolfe, & Spitler, 2012). Some groups, especially those historically underrepresented in mathematics, however, maintained some benefits as far out as fifth grade (Clements, Sarama, Layzer, Unlu, Wolfe, et al., 2017).

Results on sustainability were more positive. We expected teachers to decrease in the fidelity in which they taught with learning trajectories after project support was discontinued. However, after two years, they increased the quality of their teaching (Clements, Sarama, Wolfe, & Spitler, 2015). Recently we find even more positive effects six years after support from the project had ceased (Sarama, Clements, Wolfe, & Spitler, 2016). The largest predictor of higher fidelity years out was child gain—teachers sustain and increase the quality of teaching when they observe their children learning along learning trajectories.

EARLY ARITHMETIC: THE CAMTE APPROACH TO TEACHER EDUCATION

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As more and more young children attend preschool, it is up to the teacher to provide opportunities for children to engage with mathematics. Yet, all too often, preschool teachers receive little or no preparation for teaching mathematics to young children (Ginsburg, Lee, & Boyd, 2008). Thus, professional development for practicing teachers, that will promote both their knowledge and self-efficacy for teaching mathematics, is important.

THEORETICAL APPROACH

For the past several years (with support from The Israel Science Foundation grant No. 654/10), we have been investigating preschool teachers' knowledge and self-efficacy for teaching mathematics, as well as providing professional development based on our findings (e.g., Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2014). The theoretical framework we developed, called the Cognitive Affective Mathematics Teacher Education (CAMTE) framework (see Figure 1), takes into consideration both subject-matter knowledge (SMK) and pedagogical content knowledge (PCK) (Shulman, 1986) as well as related self-efficacy beliefs (Bandura, 1986).

Regarding teachers' SMK, we further differentiate between being able to produce solutions, strategies, and explanations and being able to evaluate given solutions, strategies, and explanations. In line with Ball and her colleagues (Ball, Thames, & Phelps, 2008), the framework also differentiates between two aspects of PCK: knowledge of content and students (KCS) and knowledge of content and teaching (KCT). Within KCT, we specifically focus on teachers' knowledge of designing and implementing tasks.

	Subject-matter		Pedagogical-content	
	Solving	Evaluating	Students	Tasks
Knowledge	Cell 1: Producing solutions	Cell 2: Evaluating solutions	Cell 3: Knowledge of students' conceptions	Cell 4: Designing and implementing tasks
Self-efficacy	Cell 5: Mathematics self-efficacy related to producing solutions	Cell 6: Mathematics self-efficacy related to evaluating solutions	Cell 7: Pedagogical-mathematics self-efficacy related to children's conceptions	Cell 8: Pedagogical-mathematics self-efficacy related to designing and implementing tasks

Figure 1: The CAMTE Framework

Our approach to early mathematics learning is one of playful learning (Hirsh-Pasek, Golinkoff, Berk, & Singer, 2009) that includes activities which are flexible, where the child is active, but where an adult aims a child toward a specific knowledge. Several principles guide our design of mathematical activities. First, the mathematical concepts and competencies embedded in the tasks stem from the mandatory Israel National Mathematics Preschool Curriculum (INMPC, 2008). Second, a series of tasks may be conceived as a single object, but they should be varied in such a way as to highlight separate competencies. For example, when working with preschool teachers, we differentiate between counting and enumerating skills. While a task may promote more than one competency at a time, a key principle of our design is that it should be clear which of the competencies is being targeted at each point of the task. As such, careful attention is paid to the exact wording of the instructions and questions involved in a task. A third design principle is that the objects used in the task be readily available to the preschool teacher and familiar to children. The fourth design principle is that modularity, adjustability, and extendibility are inherent to task design. For example, an enumeration task may include placing eight identical bottle caps in a row. This task can be adjusted by using a different amount of caps, or by placing the caps in a different configuration (such as in a circle with no obvious beginning or end), by varying the color of the caps, etc.

METHODOLOGICAL APPROACH

Our methodological approach to studying early arithmetic teaching takes into consi-de-ration the need to investigate a wide perspective of teachers' knowledge (e.g., by using questionnaires) as well as taking a closer look at how knowledge may be enacted in practice (e.g., by analyzing video recordings of our professional development programs and teachers working with children). We now offer an example within the context of set comparison and demonstrate how our methods for investigating teachers' knowledge is intertwined with promoting their knowledge for teaching set comparison, and how the methodology also takes into consideration contextual factors of learning mathematics.

To begin with, we exemplify how the knowledge cells of the CAMTE framework can be adopted to investigate teachers' knowledge related to enumerating items in a set and comparing sets (see Figure 2).

For preschool children, comparing the number of items in two sets is not a trivial matter (Tsamir, Tirosh, & Levenson, 2010). However, for teachers, one might ask, what is there to investigate? When we asked teachers to write down how one could tell, in general, which of two sets had a greater number of items, all of the teachers answered, "by counting." Yet, according to the preschool curriculum, children ought to be exposed to several ways of comparing amounts of items, including one-to-one correspondence. In order to challenge the teachers and investigate both aspects of teachers' SMK according to the CAMTE framework, we presented the teachers with several examples of sets to compare (see Figure 3 for two examples). Teachers were asked to state how they solved the problem, if there was only one way to solve the problem, and if one method of comparing sets might be more efficient than another?

Subject-matter knowledge		Pedagogical-content knowledge	
Solving	Evaluating	Students	Tasks
Cell 1: Compare the number of elements in two sets using a variety of strategies; enumerate the following large collection of items using a variety of strategies.	Cell 2: Evaluate the following strategies for comparing the number of elements in two sets; evaluate the following justification for why one set has more elements than another set.	Cell 3: What are children's common mistakes related to the cardinality principle? What are some of the common ways children compare set items?	Cell 4: Which tasks have the potential to foster children's acceptance of the one-to-one principle necessary for enumerating? Which tasks have the potential to assess children's enumerating skills?

Figure 2: The CAMTE framework and teaching number concepts

For example, in the second example, one could count the number of green and red pills, or simply and immediately state that they are equivalent by virtue of one-to-one correspondence. These activities were meant only for the teachers, and were not to be implemented with children. We specifically distanced the teachers from the preschool context, in order to investigate their knowledge in a respectful manner, while enabling them to feel some of the difficulties that some children might experience when asked to compare sets. It also allowed them to see how responses to arithmetic tasks might be influenced by the context of the problem.

Here are two sets A and B: $A = \{1, t, \alpha\}$ $B = \{7, w\}$ Is the number of elements in sets A and B equal? Yes / No How did you reach this conclusion?	Dan was ill. The doctor prescribed one green tablet every 3 hours for the first week. Then, in the second week he was ordered to take a red pill every 3 hours. Are the number of green pills and red pills equal? Yes / No How did you reach this conclusion?

Figure 3: Compare the number of items in each set

In order to investigate teachers' knowledge of children's conceptions and their knowledge of tasks, we asked teachers to design set comparison activities for their preschool children (ages 4-6 years), implement them in their preschools, and then present their findings to the group. Issues raised included the number of items to present in each set and the types of items in each set. For example, teachers planned a task where children were requested to create equivalent sets from inequivalent sets. They discussed whether the context should be sets of chocolate bars or trees with autumn leaves. How might this context affect children's ways of solving the problem? Most children like chocolate. If they think someone else has more, it would seem natural to take from the other person in order to ''even out'' the treats. However, in the matter of transferring leaves from one tree to another, it is not for sure that the children would see this as an option. Thus, in the case of leaves, children may even out the sets by removing leaves, but not by transferring leaves.

After implementing the tasks, teachers reported back to the group and to the researchers, and shared their experiences. Notably, teachers became aware of subtle differences in tasks and how these differences were reflected in the children's work. Teachers were able to see and compare how different children solved the same problem as well as what some of the obstacles were. To summarize, enhancing teachers' SMK and PCK was done mostly through their engagement with tasks. Tasks were then implemented by the teachers with their children. Thus, teachers were taught in the same way they were expected to teach in preschool. The enactment of the tasks by the teachers in their own kindergarten then served as a basis for reflection which further informed teachers' SMK as well as their PCK. Learning and practice informed each other.

MEDIATING PRIMARY MATHEMATICS FROM THE EVERYDAY TO THE SCIENTIFIC: THE STRUCTURING NUMBER STARTERS PROJECT

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Our linked research and development work in early arithmetic has been located, over the last seven years, in a South African context of very low performance in mathematics, with ongoing and widespread use of concrete unit counting. However, in this period, there have also been successes in achieving near universal access to primary education, including recent broadening of access to Grade R provision (children aged 5), and roll out of national workbooks and slowly increasing availability of resources like 100 charts and abaci. Access to pre-school remains a rarity on the ground and instructional organization is complex with learning in the early grades occurring in classes taught in one of the 11 recognized national languages. Additionally, large classes, overcrowded classrooms and developmental delays related to early childhood poverty all feature commonly in the national terrain.

Working in this context, baseline data that we gathered in 2011 indicated that nearly three quarters of cross attainment Grade 2 students were using unit count-based 'count-all' strategies for additive relations problem solving in the 1-20 number range. By 2014, having introduced tasks, resources and professional development workshops in the interim period, this proportion, in our data set, had dropped to less than half of the Grade 2 cohort. Here we report on the theoretical and developmental approaches that we have employed in order to promote these moves from counting to structuring.

THEORETICAL APPROACH

In our current work, we have adopted a 'straight for structuring' approach with Grades R and 1, in which activities that direct attention to number relations and structure are used from the start alongside, and parallel to, teachers' work with their classes on learning to count. In this approach, we view learning to count in terms of 'social knowledge', in Piagetian terms, that children need to acquire in order to work with number. However, in a context of increasing provision in schools of 'structured' resources, there is growing evidence that through this provision teachers are becoming more familiar with structured and relational views of number. We have thus moved away from neo-Piagetian approaches to progression that see increasingly efficient counting as leading eventually into structure (e.g. Steffe, 1983; Wright, Stanger, Stafford, & Martland, 2006) towards a view based more on Davydov (1990) attention to structure from the start, a view where early number and arithmetic are seen as part of a network of scientific concepts in the Vygotskian (1986) sense, with everyday knowledge relating to counting being gained alongside instruction focused on number relations and structure.

Alongside this we also hold to a central theoretical position arising from adapting Vygotsky's view of 'method' as being both tool and result:

> The search for method becomes one of the most important problems of the entire enterprise of understanding the uniquely human forms of psychological activity. In this case, the method is simultaneously prerequisite and product, the tool and result of the study. (Vygotsky, 1978, p.65).

Although Vygotsky was originally referring to 'method' as the approaches taken within the discipline of psychology, we attach his argument that the application of any method is not independent of the results of that application to the study of pedagogy. Seeing tool-and-result as dialectically connected means that teaching ''tools' (pedagogy) cannot be treated as independent of the 'results' (student learning), but instead that both the form and content of pedagogy are inextricably bound up with the learning outcomes. From this position, learning is not so much 'situated' within particular pedagogical approaches, nor is learning simply 'mediated' through classroom practices. Rather learning is dialectically bound up with the chosen pedagogical methods. Pedagogy is not simply something applied to an already existing body of knowledge and then carried out in lessons as a means of transmitting that body of knowledge to the students. Pedagogy and learning are mutually co-constructive.

From these overarching theoretical positions we have developed a framework for examining the mediation of primary mathematics—the Mediating Primary Mathematics (MPM) framework—that allows us to examine the mediating means through which teachers might work with young learners on the structuring of early number and so move away from a reliance on the everyday knowledge of learners—primarily unit counting—to inducting learners into mathematics as a structured ‘scientific’ discipline (Vygotsky, 1987). The MPM framework is both theoretically and empirically derived and driven. The theoretical foundations of the MPM framework are located in three linked positions on teaching and learning mathematics, all rooted in sociocultural theory. First, that pedagogies that link teaching and learning mathematics are operationalized through the tasks that teachers present to learners in class and example spaces selected to be worked on within the tasks, and the subsequent activities (physical and mental) that learners engage in. Second, that these instructional tasks and example spaces provide the ‘raw materials’ that together with the mediational means of artefacts, inscriptions and talk both underpin teachers’ attempts to foster learning and learners’ activities, The mediating means themselves do not determine what mathematical objects are brought into being in lessons—how the teacher works with them and where she directs learner attention are determining factors. Fingers, for example, can be drawn on by teachers as artefacts that continue to promote counting procedures: we are particularly interested in the way mediating means may be brought into play with a focus on structure, for example, using fingers to find and explore the structural composition of, say, seven as five and two or six and one rather than simply as the result of counting out seven fingers singly. Third, such leveraging of example spaces to focus on structure and generality requires connections to be made in and between artefacts, inscriptions and the nature of talk.

The empirical grounding of the framework lies in using it to analyze videos of lessons. In the development of the framework we employed a cyclical and iterative interplay between the theoretical and empirical, through repeated cycles of developing our theoretical view of mediation, using this to check if it could be applied to lessons, re-framing the theory and the MPM before again applying it for analytical purposes (for a full description of the development of the MPM framework, see Venkat and Askew, 2018).

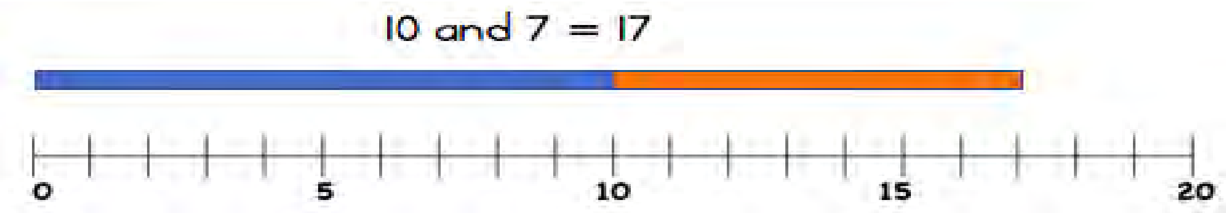
Having thus established a stable framework, we have used it to examine teachers’ current practices (Venkat et al, under review). But we are also using it now to inform the design of pedagogical tasks, and it is that that we examine here by looking at one strand of our research and professional development work: the Structuring Number Starters (SNS) project. It is a curriculum expectation that early year teachers spend ‘at least 20 minutes per day at the start of the Mathematics lesson’ focusing on ‘mental mathematics, consolidation of concepts and allocation of independent activities’ (Department for Basic Education, p.11). Our baseline observation data indicated that many of our teachers had interpreted ‘mental mathematics’ as choral class counting and rote recall of number bonds: the SNS project engages the teachers, through workshops and classroom coaching, with thinking about and applying a broader range of tasks and examples that might develop both their breadth of understanding of what comprises 'mental mathematics' and also support learners in moving from naïve counting approaches to thinking about mathematical structure.

METHODOLOGICAL APPROACH: DESIGNING AND IMPLEMENTING SNS TASKS

Task design

A key part of the SNS project approach is the development of tasks and examples that attend to both structure and connection in the context of early number. Structure, for us, involves tasks that require awareness of relationships between numerical quantities, rather than seeing quantities only as counted entities. Connections, following the work of Askew et al. (1999), involves seeing both mathematical processes and results as located in networks of connected ideas, and following Watson and Mason (2006), the interconnecting of examples facilitates attention to structure and generality through patterns of variation and invariance.

An example of a task building on this kind of approach is presented below, beginning with an example that our Grade 2 learner assessments indicated that large numbers of children could answer:



With teachers, we suggest following up with a series of tasks connected to this one, and building in variation across a range of dimensions, for example: What can we say about 7 and 10? What is 10 more than 7? What is 7 less than 17? We also connect these word formats to number sentence formats, and discuss whether, e.g. 10 + 7 = [_], 7 + [_] = 17 and [_] + 10 = 17 should be seen as three separate problems, or three connected problems, with implications for whether some of these problems are seen as easier/harder than others, or all of similar levels of difficulty. We then focus on adaptive variations, based again on using number relations to reason about answers, rather than viewing subsequent tasks as separate calculations to start afresh: e.g. If 10 + 7 = 17, what can we say about 9 + 7?

Task implementation

Resources supporting attention to number structure and relations, like the semi-structured number line illustrated above marked in 5s, are distributed alongside the tasks/examples. We work with the teachers on variations in wording that have to be communicated in instructional talk, and explore variations in task format. These mediating forms both expand the example spaces that children encounter and also develop the range of learner activity, helping both teachers and learners to come to see examples as connected, thus supporting moves towards working with mathematical relationships and processes in generalized rather than localized ways. Following termly workshops with teachers in participating schools discussing such tasks, team members go into classes and observe and co-teach these tasks in classrooms.

GOING FORWARD

Our observations point to gradual changes in teachers' take-up of attention to numerical structure in their pedagogic practices. In a context of the near-endemic evidence of counting-based approaches well into the middle grades (Schollar, 2008), supporting moves towards attention to number as a structured system in which this structure can be leveraged for efficiency and so form the basis for subsequent learning represents a cultural shift rather than simply an alternative pedagogical approach. Our results suggest that, in this ground, we are making progress in the early number ranges.

Taking the work forward involves supporting teachers to connect the emerging attention to structure in the context of early number into their work in number ranges beyond 1-20. This brings us back to Vygotsky's tools-and-results argument. If teachers' methods and tools for working with number in larger number ranges tend to sideline attention to structure, either through allowing ongoing counting, or through focus on traditional algorithms accompanied by talk that ignores numerical relationships, then the outcomes seen at learner performance level are unsurprising. Our current attention is therefore to the design of tasks and pedagogical strategies focused on building connections between early number structure and use of this structure in higher number ranges.

SPONTANEOUS FOCUSING ON NUMEROSITY IN THE DEVELOPMENT AND ENHANCEMENT OF EARLY NUMERACY

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Noticing of mathematics all around you is an essential part of mathematical development from early on. This presentation reviews research investigating children's Spontaneous Focusing On Numerosity (SFON) and considers the role it might play in the development of counting and arithmetical skills. SFON refers to a process of spontaneously (i.e., not prompted by others) focusing attention on the exact number of a set of items or incidents. This attentional process triggers exact number recognition and using the recognized exact number in action (Hannula & Lehtinen, 2005). The presentation describes how SFON tendency can be assessed and enhanced, and suggests the measures of it to be indicators of the amount of a child's self-initiated practice in using exact enumeration in his or her natural surroundings.

THE NEED FOR DEVELOPMENT OF EARLY MATHEMATICS EDUCATION

There is great need for theoretically based programs that make use of children's typical ways of learning in everyday surroundings. A recent US National Research Council report (Cross, Woods, & Schweingruber, 2009) describes the teacher's crucial role in expecting and supporting children's ability to make meaning and mathematize the real world, as well as creating a nurturing and helping mathematics talk community (Fuson, Clements & Sarama, 2015). Problematically, many preschool and school mathematics materials - particularly for low-achieving children - are often characterized by quite meaninglessness and rote-based learning. Many early educators do not know how early mathematical thinking could be systematically supported in any other ways, and they often have negative attitudes about early mathematics education (Hachey, 2013). Sadly, still, many existing teacher support materials for early mathematics education focus on listing separate early numeracy skills of children, which are often encouraged to be taught by traditional formal exercises with very little support for using the skills in everyday life of children. Typical, 1st grade mathematics textbooks seem to have the same limitation. There are extremely few tasks for mathematizing the real world outside of the textbook.

This kind of approach fails to make use of children's spontaneous activities and play, even though those children whose early mathematical skills develop the best, more often practice and use their mathematical skills in these situations (Hannula & Lehtinen, 2005). It would be beneficial to get early educators to help all children to become aware of the mathematical aspects of these kinds of informal activities, so that all children would get enough versatile mathematical practice. This kind of spontaneous action of children is crucial for numeracy development. Children differ in their realizing of mathematical aspects in their surroundings, and supporting focusing on numerical aspects is an efficient way of promoting early numeracy development (for a review, Hannula-Sormunen, 2014).

MAIN FINDINGS OF STUDIES ON SPONTANEOUS FOCUSING ON NUMEROSITY (SFON)

Learning to focus attention on the aspect of number in one's surroundings is one of the significant elements of early mathematical development, leading to the necessary practice in utilizing number recognition processes in one's action. The studies from a variety of countries show that SFON tendency in early childhood is positively and domain specifically related to the development of numerical skills up to the end of primary school (e.g. Batchelor, Gilmore & Inglish, 2014; Edens & Potter, 2013; Hannula, 2005; Hannula & Lehtinen, 2005; Hannula, Lepola & Lehtinen, 2010; Hannula, Räsänen & Lehtinen, 2007; Hannula-Sormunen, 2014; Hannula-Sormunen, Lehtinen & Räsänen, 2015; Kucian et al., 2012; Nanu et al., 2018). These studies reveal that some children's world may appear as being full of opportunities to use their mathematical skills, while some others do not seem to notice the mathematical features of the environment at all. Individual differences in this separate attentional disposition named "Spontaneous Focusing On Numerosity (SFON)" have been found to predict domain-specific differences in the development of mathematical skills throughout childhood and primary school (Hannula et al., 2010; Hannula-Sormunen et al., 2015; Nanu et al., 2018). SFON tendency impacts mathematical development through the promotion of children's self-initiated practice with related mathematical aspects of their everyday environment (Hannula & Lehtinen, 2005). SFON tendency has been shown to be a rather stable component of early mathematical skills, that is not explained by motivational factors (Edens & Potter, 2013; Nanu et al., 2018), general attentional skills, non-verbal IQ, comprehension of instructions (Hannula & Lehtinen, 2005; Hannula et al., 2010), or inhibition (Nanu et al., 2018). Importantly, there is already some evidence that enhancement of SFON tendency can be done at day care or in preschool, and that this enhancement leads to the subsequent development of children's numerical skills (Hannula, Mattinen & Lehtinen, 2005; Hannula-Sormunen et al., 2016; Mattinen, 2006).

METHODS FOR ASSESSING SFON

In one of our first SFON tasks, the experimenter was feeding a toy bird with small numbers of berries and she asked the child just to "do the same" without giving any further instructions (Hannula & Lehtinen, 2005). It appeared that this simple activity revealed substantial individual differences in four-year-old children's spontaneous focusing on numerosity. Some children immediately noticed how many berries the experimenter gave to the bird, while others focused only on other aspects of the activity, but not the number of berries. Since then, more than two dozen different behavioural tasks and observational methods have been designed for SFON (Hannula-Sormunen, 2014). The aim of SFON assessments is to obtain a reliable indicator of a child's general SFON tendency across different task contexts. This kind of measure is aimed at capturing the amount of a child's self-initiated (i.e., not prompted by anyone else, in the situation) focusing on numerosity, and thus the amount of practice acquired in utilizing enumeration skills in his or her surroundings (Hannula & Lehtinen, 2005).

In order to enable the measuring of children's spontaneous behaviour the tasks must be novel to the child, and they can have only very few trials. Furthermore, when presenting SFON tasks, no phrases which would reveal that the task is somehow mathematical or quantitative can be used. Neither can the experimenter give any feedback during the task performance. This way the testing of SFON does not limit the ways how the child could interpret which are the relevant aspects of the task. Each SFON task trial is presented only when the experimenter has got the child's full attention on the task, so that the child's level of general attention, or task motivation, does not explain the individual differences in SFON. Critically, in order to hinder the confounding effect of number recognition skills on the measure of SFON, the SFON task can include only so small numbers of items or incidents, which all participants should be able to recognize. Moreover, all other cognitive requirements of the SFON tasks need to be at manageable level for all participants, so that participants' insufficient motor skills, inhibition, verbal production, working memory do not explain individual differences in the SFON tasks (Hannula, 2005, Nanu et al. 2018).

Guided Focusing On Numerosity (GFON) task versions have showed that the children who had no SFON responses in the original SFON tasks are able to deal with the cognitive task requirements after their focus has been explicitly guided to the numerical aspects of the SFON task (Hannula & Lehtinen, 2005, Hannula et al., 2010). Children's performance on the guided tasks supports the idea that SFON is dissociable from other sub-processes of utilizing exact number recognition in action. The analyses of video-recorded performance in the SFON tasks allow signs of enumeration acts and indications of children's understanding of the quantitative goal of the task being recognized as SFON. The use of these methods allowed Hannula and Lehtinen (2005) to separate the attentional process called SFON, which is defined as a process of spontaneously focusing attention on the exact number of a set of items or incidents and using of exact numerosity in one's action.

SFON-BASED INTERVENTIONS

Finally, main principles of SFON-based early numeracy programs are described. These interventions have been based on the idea that early educators should be challenged to ask questions that stimulate further mathematical thinking among children (Saebbe & Mosvold, 2016) as part of practically any everyday activity with the children (Hannula, Mattinen & Lehtinen, 2005; Mattinen, 2006). Björklund and Barendregt, (2016) show that even though preschool teachers regularly engage children in communication about mathematical phenomena, they do not systematically use the physical environment as a point of departure for directing children's attention to specific mathematical concepts or principles. A related, challenging aspect of previous SFON studies was recognized by Mattinen's study (2006), which demonstrated how easily early educators go along with the children who spontaneously focus on numerosity and provide these children with more and more numerical activities, while the children who do not spontaneously focus on numerosity are easily left without guidance towards numerical activities. To counteract this, the SFON-based interventions can be used to model the desired SFON behaviors leading to children's own self-initiated practice of enumeration and other early numerical skills.

> Somehow I think, that small children have not been even offered any maths at all. It [mathematics] has been only during the kindergarten year, and then the numbers and write number seven - - To me a great novel notion has been that this everyday surroundings and how maths is present in it in every moment. If you think that this learning environment has been here, then this is my realization. And also children have realized this. Maybe at some

point, when maths was so much present in everyday situations, one realized that this is how it goes.

This is a quotation from one pilot intervention study for 3–5-year-old children aimed at enhancing early numeracy as part of early education in day care (Mattinen et al. 2010). In this example, the early educator describes her old and new ways of thinking about early math and emphasizes how mathematical skills really can be guided through everyday activities. She thinks mathematical thinking can be part of all situations where adults and children interact. The integration of SFON and other early mathematical skill training (e.g., Hannula-Sormunen, Alanen, McMullen & Lehtinen, 2016) has proved to be a fruitful approach both from the points of view of early educators' professional development and participating children's mathematical development.

TO EXPERIENCE THE FIRST TEN NUMBERS AS NECESSARY GROUND FOR ARITHMETIC SKILLS – THROUGH THE USE OF FINGER PATTERNS

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In the FASETT project we propose that to develop skills for arithmetic problem solving, children need to experience specific aspects of numbers and operations, in particular numbers' part-part-whole relations. In an intervention program conducted in Swedish preschools we developed and tested activities and artefacts that specifically direct children's attention towards necessary aspects by using finger patterns. Following the children's ways of experiencing and operating with numbers give support to the conjecture and theory, that children develop their number knowledge and learn arithmetic skills experientially, and finger patterns may work as facilitating means in the learning process.

THEORETICAL FRAMEWORK: LEARNING AS EXPERIENCING

The theoretical framework in the FASETT-project—variation theory—stems from phenomenography, a research approach studying the different meanings things have for people (Marton, 2015; Marton & Booth, 1997). Actions are related to how we experience a situation or a problem. Thus, how we act is in accordance with how we see or experience. So, to act in a different way one must see something in another way. Furthermore, how something is seen is due to what aspects of that something we attend to at the same time. If 'the same thing' is seen differently by different people, it is likely that they discern different (or partly) different aspects. From this follows that, to develop for instance number sense, certain aspects are critical and of decisive significance to discern. This presumption is under-pinning the project; There are certain aspects of numbers and arithmetic operations that must be discerned, for instance, to solve problems like 7= _+2. At a certain point in time some children have discerned these critical aspects, whereas other have not. To identify what children need to discern, by closely studying how they experience numbers and arithmetic problems, is one leg this project rests on. The other is to help them to experience numbers and operations in a different way, by offering activities and tasks that opens up for possibilities to discern aspects previously not discerned.

Variation theory states that learning is a function of discernment and discernment comes from experiencing variation. In other words, that what varies against an invariant back-ground, is likely discerned. So, for instance, to provide opportunities to discern critical aspects of relations within numbers, it is necessary to open up for a variation of those relations. Consequently, activities and tasks must be designed by principles of variation and in-variance in a way that draws the learners' attention to the targeted aspect. Thus, from a variation theory perspective, it is not just the character of activity or task as such, for instance whether it is explorative or engaging or not, that is of interest. Instead, what is of significance is how the critical aspects come to the fore by the means of patterns of variation (Kullberg, Runesson Kempe & Marton, 2017; Watson & Mason, 2006). To summarize, The FASETT-project takes the point of departure in a theory where learning is seen as experiential that is, 'learning as seeing'. This have consequences for how we approached to study children's developing ways of experiencing and operating with numbers.

METHODS TO CREATE OPPORTUNITIES TO EXPERIENCE THE TEN FIRST NUMBERS BY MEANS OF FINGER PATTERNS

From interviews with 103 five-year-old children we gained knowledge of their experiencing numbers, using fingers and early arithmetic skills. For instance, we found that some children thought six could only be represented by one whole hand and the thumb on the other hand. Hence, to develop number knowledge and arithmetic skills it was critical for them to experience that six could be represented in different ways and constitute different combinations of smaller sets. Moreover, some children could count from one to ten, or further, but did not know the cardinal value of six. For them it was also critical to experience that numbers can be represented with similar number of fingers (and other items).

Earlier research has shown that fingers can be used in different ways as aid in solving arithmetic tasks; either to keep track of counted numbers (Fuson, 1988) or to represent sets in a part-whole relation (Brissiaud, 1992). The latter was found to support the theoretical conjecture of ours, how children learn to experience numbers' "manyness". Altogether, we identified three critical aspects for the participating children to learn to use a structural approach in arithmetic tasks (Table 1). Furthermore, when these aspects were brought attention to, two other aspects of arithmetic tasks emerged, which enabled the children fluency and proficiency in solving the tasks: *the invers relation between addition and subtraction*, and *commutativity* (the ordering of addends).

To experience that…	…the child has to have experienced that:
numbers can be represented by fingers	numbers can be represented in *different* ways
numbers can be represented by finger patterns	fingers can be represented by *different* finger patterns
numbers can be structured as a part-part-whole relation	numbers can be structured as *different* part-whole relations

Table 1. Aspects necessary to learn to use finger patterns to represent numbers (left column) and patterns of variation that enable children to discern the aspects (right column).

We designed an intervention programme based on variation theory principles that was offering children opportunities to experience necessary aspects of numbers through carefully designed patterns of variation (see Table 1). We also based our programme on Neuman's (1987; 2013) conjecture that finger patterns can help children to discern necessary aspects of numbers, such as the part-part-whole relation. Neuman advocated that seeing numbers as finger patterns could facilitate discernment of the structure of numbers, for example seeing a number as a whole that may contain two or more parts where in particular the whole hand seen as a comprehensive "five" would support children in extending their subitizing range (c.f. conceptual subitizing, Clements, 1999). But to experience the flexible use of finger patterns, the children first had to experience numbers as composed sets of units that can be represented with fingers without counting.

To enable the children to discern critical aspects, we designed activities and artefacts in which the aspect in question would vary and other aspects would be held in-variant. Our main goal was to enable the children to see numbers as a part-part-whole relation. To accomplish this, we used finger patterns as means to structure numbers and visualize both parts and the whole simultaneously (c.f. Brissiaud, 1992). The preschool teachers used variation theory e.g., when sequencing tasks in the activities, or to make contrasts between different finger patterns (six [nvariant] can be recognized [vary] as both *three* fingers on one hand and *three* on the other, as well as *two* fingers on one hand and *four* on the other), in order to make the children notice the critical aspect that a number e.g. six can be represented in different ways.

One example of an activity designed on variation principles, the ten-snake game (Picture 1), was enacted to enable children to experience the structure of numbers that is a part-part-whole relation. The game started with each child first modelling the whole (10) with fingers. Thereafter the teacher covered one part (4) and the children were asked to show with a finger pattern the numbers of beads that were shown (6). By looking at their fingers the children could find out (see) the covered part (4 as a finger pattern) without having to rely on counting the fingers. The finger patterns illustrate the known part (6 as a finger pattern) and the unknown part (4 as a finger pattern). Some children turned their hand to see the four whereas other children only looked at their knuckles. During the game the teacher directed the children's attention towards the whole (invariant) and the parts that varied (e.g., 10-6=4 and 10-4=6, 10-3=7 and 10-7=3 etc.)

Picture 1: Finger pattern illustrating parts of a whole.

One premise for the intervention programme was the close collaboration between teachers and researchers, in that the activities were documented and refined during the programme, to best face the children's individual need for certain aspects to be foregrounded. The process is best described as iterative. Teachers' enactment of the same activity with different groups of children was video recorded, in order for the researchers and teachers to be able to discuss the enactment of the activities. This became an important tool in the process of the intervention since it gave an opportunity to discuss children's learning, the refinement of the implementation of critical aspects in the activities, and the teachers' use of the theory.

IMPLICATIONS FOR PRESCHOOL EDUCATION

The aspects that were found to be necessary for developing number knowledge and arithmetic skills are not new. Baroody (2016) and in particular Schmittau (2004) and Davydov (1982) also emphasize the necessity to operate with part-whole relations to solve arithmetic tasks and many observations have been documented of children's use of fingers in solving arithmetic tasks (Fuson, 1988, Baroody, 1987) to invent shortcuts or keep track of counted units. But the way to systematically enable young children to *experience* numbers is a novel approach and particularly in the Swedish preschool practice where learning goals are very broad and the national curriculum (National Agency for Education, 2011) is not giving guidance for how to teach mathematics other than catching children's attention to mathematics in play and every-day situations. Our approach directs attention to children's ways of experiencing numbers and how to enable children to discern what they have not been able to experience before. The project thus aims to provide the field of early childhood mathematics a pedagogy based on a solid theoretical framework that embrace the individual learners' perspective. This is of importance within the age group we direct our attention to, because of the large variety of ways to experience numbers that we observe among preschool children. Attention to children's ways of experiencing numbers, and knowledge of aspects that are necessary for developing arithmetical skills gives preschool teachers pedagogical tools to facilitate children's arithmetical development.

CONCLUDING THE EFFICIACY OF THE VARIATION THEORY APPROACH

Rather than constructing a trajectory of strategies that are to be learnt in order to solve arithmetic tasks, we propose a pattern of ways to experience the numbers in a task. Our framework is experiential, meaning that numbers have to be experienced in certain ways that are decisive for what is possible to do with numbers. Different arithmetic tasks demand different aspects to be discerned, or in a teaching act, to be offered to experience through a pattern of variation. What strategy to use to solve an arithmetic task is thus, according to our conjecture, not relying on learnt strategies or procedures, but on how the child has learnt to experience or "see" the task at hand.

Evidence of the efficacy of this approach is shown in the children's qualitatively different ways of encountering novel arithmetic tasks, where in particular the use of fingers as means to structure numbers were found to be a strong facilitator, not least to those children who initially did not experience numbers as units possible to operate with. The frequency of correct answers in follow-up interviews with the children support our claim, however, our contribution to the field of research is primarily the approach to describe learning as developing ways of experiencing numbers.

CONCLUSIONS FROM THE RF AND LOOKING FORWARD

The five research groups here presenting their contributions regarding theoretical and methodological advancements in the field of arithmetic learning and teaching mirror the broad spectrum of research approaches that constitute this field. The large-scale interventions building on the state of the art research provide solid evidence of the impact research based education have. A contrasting, but not contradicting approach is found in projects that take departuring point in specific theoretical frameworks, implementing and re-fining specific epistemological views in designed teaching activities. The latter may contribute with a comprehensive understanding of why research based interventions work, but also how contextual aspects influence on children's learning opportunities that demographic data alone cannot explain.

One of our main questions to elaborate upon in this RF was the issue of methodology. All of our participating research groups have invented innovative methods and designs for studying the learning or teaching acts. What consolidates our different approaches is directed attention towards conceptual understanding that is facilitated through awareness and attention to aspects of arithmetic skills that promote development. In particular, we share a common focus on the youngest learners in the education system, to which numerical notions and arithmetic principles are novel or emerging. Our combined efforts in research during the last decade have proven that arithmetic development cannot be taken for granted and that advanced skills rely, and to many children are bound, to the preschool and primary school teachers' professional work and skills.

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VITAL SIGNS OF COLLECTIVE LIFE IN THE CLASSROOM

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For almost two decades, the researchers in this Research Forum have been developing strategies to represent and model collective dynamics at the classroom level. However, efforts to do so with singular visual, auditory, or textual representations have not been particularly fecund. A possible reason for limited success is the sheer complexity of collective human phenomena. In this forum, we share our current work as a re-searching collective, explore past methodological perspectives, and present new possibilities afforded by the metaphor "vital signs," as applied to classroom episodes.

INTRODUCTION

Within mathematics education research, there is an extensive body of literature pointing to group dynamics and discourse (see Francisco, 2013). Indeed, the 1990s and the following two decades were ripe with visions of cooperative learning, collaborative inquiry, and communities of practice (Bauersfeld, 1995; McCrone, 2005; Yackel & Cobb, 1996). This work helped to transform mathematics classrooms from students sitting isolated and in silence to students working cooperatively in groups. Further, this research led to an emergence of efforts to understand, describe, and define collective systems within the field of education generally, and was taken up specifically in mathematics education (e.g., Bowers & Nickerson, 2001; Goos, 2004; Saxe, 2002; Sfard & Kieran, 2001). From this work came a growing realization that acts of cognition often arose from the classroom as a whole and could not be traced back to any one individual. However, in spite of this realization, researchers descrybing and analyzing group interaction often continued to report findings based on an individual's understanding or achievement results. Even when group work was considered as a whole, the methodological tools available (e.g., video recording, transcribing, teacher and student artifacts, etc.) were often limited to analyzing individual contributions of actions and utterances, rather than to the collective as one unit.

The actions of teachers are often in response to the classroom collective (Davis & Simmt, 2003; Cobb, 1999). That is, experienced educators are able to observe overall classroom activity and say something about the nature and vitality of the learning.

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However, what is it that they are attending to? For approximately two decades, the contributors to this forum have engaged in researching collective action in the mathematics classrooms (e.g., Davis & Simmt, 2003, 2006; Martin & Towers, 2003, 2015; McGarvey & Thom, 2010; Proulx, Simmt & Towers, 2009; Thom & Glanfield, in press). Grounded in diverse yet complementary frameworks such as complexity science, network theory, embodied cognition, and enactivism, our previous work purposefully stepped away from the analysis of mathematical behaviour of individual agents and attempted to conceptualize the groups or an entire classroom as one collective agent. In referring to collective systems, we point to the kinds of interactions that may emerge in and for the group when learners work together on a joint mathematics project, problem, or task. For the past three years, we have come together to systema-tically work on the following mutual concern: *How might we observe, document, display, and analyze data from a collective learning system as one learner?* Through this work we have found the metaphor of "vital signs" to be particularly useful in guiding our work. In a similar manner to how health professionals monitor the health of bodily systems, we wish to explore the ways in which a classroom system might also have *multiple vital signs*; that is, when monitored in combination, the vital signs afford robust insight into its systemic viability. We propose that a suite of "classroom vital signs" could be used to distinguish between different forms of collective classroom activity while pointing to critical elements of dynamic engagement.

The four papers in this research forum build on this work to examine the potential for conceiving classroom collectives as adaptive and self-maintaining complex systems. The papers introduce complex systems as an interpretive frame and identify the potential this framework holds for observing and analyzing classroom-based research in mathematics education. Together, we share our research team's efforts to develop methodological tools for observing classrooms as collective systems under the metaphor of "vital signs." In the first vital sign, *utterance length and distribution*, McGarvey and Simmt examine the frequency and number of words spoken by teachers and students in a lesson to see what insights into classroom discourse patterns become visible. The second vital sign, *(non)actions on the board*, Mgombelo, Thom, Glanfield, and McGarvey focus their attention on the most common of classroom teaching tools—the board. What might we learn about collective activity by the writing on, pointing to, and erasing of chalk or marker on the board? Third, the *emergent ideational network*, by Proulx and others, examines the structure of ideas emerging over the course of a lesson and what it reveals of collective engagement. Finally, Glanfield and Thom's contribution of *Pirie-Kieren model and the dynamics of ideas*, explore collective understanding by modelling the emergence of ideas within a classroom. Individually and together, these four vital signs offer insight into classroom collectives. Our overall purpose is to consider the value of observing classrooms as collectives as well as potential methodological tools that may serve as vital signs of collective classroom life.

THEORETICAL FRAMEWORK

The four vital sign papers that follow are all rooted in complex systems research—an approach to inquiry that investigates how relationships between parts of a system can give rise to novel and unanticipated collective behaviours. Complex systems are a particular class of phenomena that include, for example, weather systems, world economies, human communication systems, nervous systems, and many others. Each complex system arises from the inextricable layering and entanglement of biological, social, societal and environmental sub-systems (Davis & Simmt, 2006, 2016). Phenomena occurring within the systems may be unpredictable in foresight, but are potentially understandable in hindsight. Complex systems present "collective possibilities that are not represented in any of the individual agents" (Davis & Simmt, 2003, p. 140). These collective possibilities are established through a self-initiating, self-organizing, and self-sustaining process. Another key aspect of complex systems is the dialectical entanglement of the system and its environment. That is, the system both shapes and is shaped by its environment. In this sense, a collective system can be defined as a complex system in which agents spontaneously interact and adapt to each other, organizing and sustaining a learning process in a collaborative and collective way. This description captures how we view mathematics classrooms as collectives and it provides the theoretical foundation for our work. Our goal is to develop methodological tools to better understand the dynamics of the classroom as a collective whole, rather than continuing to treat classroom interactions as solely a series of distinct individual contributions. While we do not discount the value of research that explores individual understanding, we recognize that teacher actions and decision making within classroom contexts are often not based on the multitude of individual actions, but on the teacher's sense of the class as a whole of which the teacher is a part (Towers, Martin & Heater, 2013). We choose to understand the whole by developing tools to help us analyze how the collective, as a coherent entity, learns.

While the literature on collective learning is predominantly theoretical, preliminary work on new techniques for analyzing and interpreting group activity received some attention in the early part of this millennium but efforts were not sustained, perhaps because individual researchers and teams did not systematically co-develop and share these techniques (e.g., Powell, Francisco & Maher, 2003; Rasmussen & Stephan, 2008). Research methodology for collective systems outside of school contexts is a burgeoning scientific field of study. In our previous work (McGarvey et al., 2015), we turned to studies that provide insight into the adaptive learning and self-organization of complex and collective systems such as insect colonies (Johnson, 2002), small-world communication networks (Watts & Strogatz, 1998), and social movements (Kilgore, 1999). These studies and many others, make use of modelling techniques to help create visual expressions of complex systems (e.g., Bender-deMoll, 2014). We believe that exploring different modelling techniques of some aspects of collective activity may be useful for observing and gaining insight into global traits and group activity of collective systems.

COMMON DATA FOR ANALYSIS

The contributors to this research forum were asked to analyze data from two classroom lessons by modelling a vital sign they had previously developed (McGarvey et. al, 2017). While modelling complex systems typically requires large data sets from multiple sources, we wondered what we might learn by modelling classrooms using the typical tools available to us as researchers: a single camera video recording and a transcript of a classroom lesson. After exploring our own data and other publically available sources, we selected two of the eighth grade TIMSS videos (see timssvideo.com): (1) Solving Inequalities (JP4) in Japan; and (2) Exponents (US3) in the United States. We chose these two lessons because we anticipated that they would reveal differences in their vital signs due to differences in culture, in the physical arrangement of the classrooms (see Figure 1), and in styles of teaching (predominantly whole class to small group instruction. Both lessons are approximately 50 minutes in length with 35 students in the Japan class and 36 students in the US class.

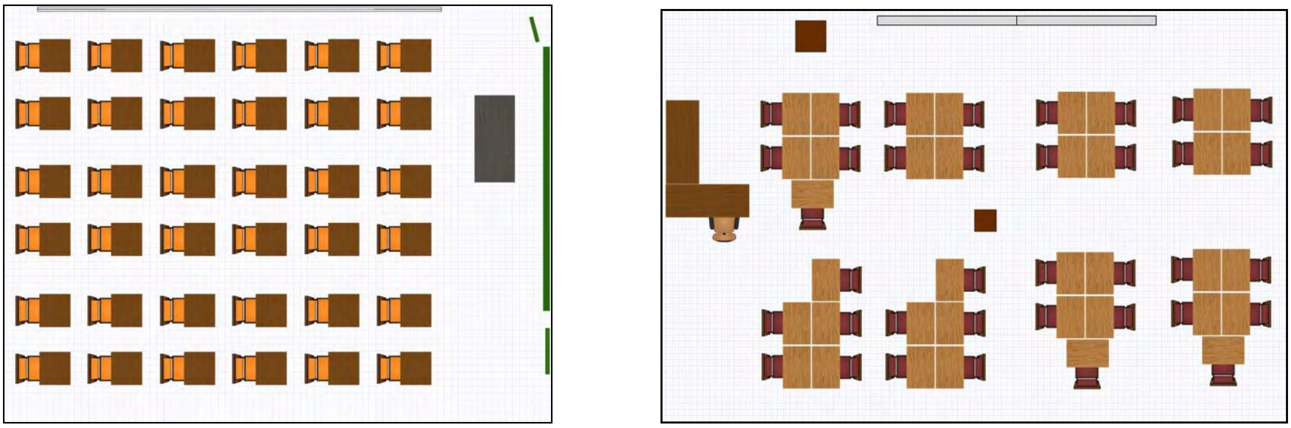

Figure 1: Japan (left) and US (right) classroom physical arrangement

The data available through the TIMSS video site included a video of the class, an English transcript, and a lesson graph outlining the timeline and lesson content. Lesson storyboards for the Japan class (Figure 2) and the US class (Figure 3) are shown below. Using the information available from the video and transcripts, the researchers modelled, analyzed, and compared the collective activity in the two lessons using their chosen vital sign. The four papers explore discourse patterns through utterance length and distribution; (non)actions on the board; the network structure of the emergence of ideas; and dynamics of ideas based on the Pirie-Kieren model.

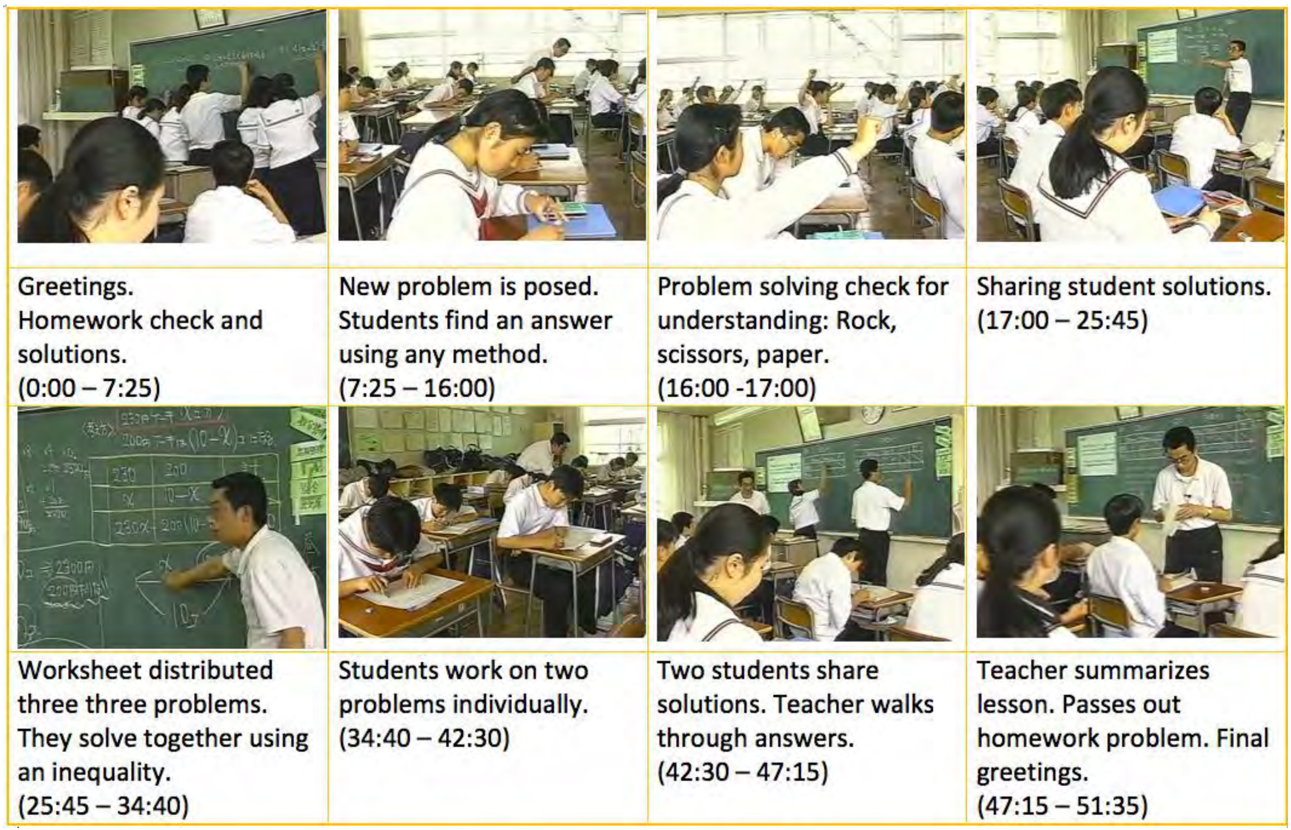

Figure 2: Japan lesson storyboard

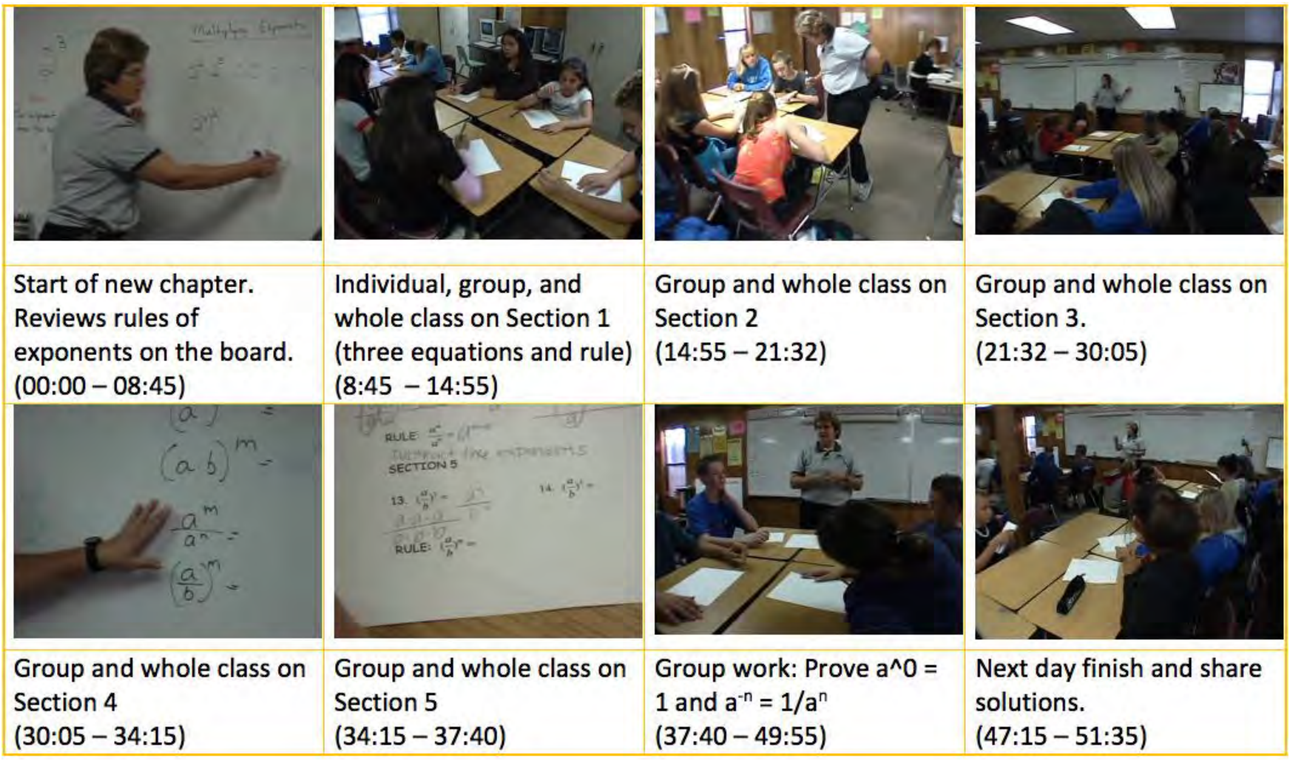

Figure 3: US lesson storyboard

VITAL SIGN 1: UTTERANCE LENGTH AND DISTRIBUTION

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INTRODUCTION

The focus on discourse has been prominent over the past few decades in mathematics education. Much of this work has attempted to delve into discourse as a way to provide access to individual cognitive processes, classroom sociomathematical norms, student engagement, types of questioning, or pedagogical content knowledge through the use of video and transcripts (see Ryve, 2011). Efforts to identify specific discursive moves such as high-level questioning, revoicing, or argumentation provide evidence indicating that discourse patterns have an impact on student learning (Hiebert & Grouws, 2007; Yackel & Cobb, 1996). As Clarke (2010) suggests, the nature of discourse in mathematics classrooms, "provides a key indicator of pedagogical principles underlying classroom practice and the theories of learning on which these principles are implicitly founded" (p. 3).

If we believe that discourse patterns matter, how might they be revealed in a vital sign without necessarily attending to the actual words, the meaning of the words, to which individuals are speaking, or to the intention of the utterance (e.g., questioning, student response)? If a classroom is a complex system, then its path is unpredictable in detail, but it may have patterns of general behaviour coinciding with the collective behaviour emerging from the interactions of its multiple bodies (Davis & Simmt, 2003). Such collective behaviour is non-linear, unpredictable in foresight, but potentially recognizable in hindsight as it feeds back onto itself and into the ongoing behaviours of the individual beings in the classroom. Examining the utterance patterns of teachers and students may provide us with insight into the classroom as a whole. Our goal here is not to say whether the discourse patterns in some classes provide better or more opportunities for learning, but that discourse patterns reveal differences in classroom collectives and may hint at implicit pedagogical principles and theories of learning in practice.

DATA AND METHOD

We chose to use the English and English translated transcripts as the primary source of our data. The transcripts provided the time of the words spoken in minutes and seconds. We also made use of the storyboards in Figures 2 and 3 to mark where large group, small group, and individual seatwork took place. In the Japan classroom, the interaction included individual seatwork, solution sharing at the board, and large group interaction. In the US classroom, the class oscillated between small group work on specific problems to large group discussions. The camera in the US classroom followed the teacher as she spoke with students during small group discussions, so the transcripts only captured the teacher to small and large group interactions, not student-to-student interactions. We wondered what might be revealed about the patterns of discourse simply by using descriptive statistics on the transcripts and general markers of class activity.

We identified each line in the transcript as being spoken either by the teacher or a student. We defined an utterance as talk uninterrupted by others, that was continuous but allowed for short pauses. For example, a teacher speaking to the whole class, pausing briefly, and continuing was considered one utterance. When there was a pause greater than 5 seconds or if another person spoke, we treated the string of words as separate utterances. Pauses were determined using the time indicators on the transcripts. We recognized that we were not accounting for overlapping talk or other nuances that could be determined by attending to content; however, our goal was for a global visual of discourse rather than a detailed account.

Once the transcripts had been coded in this way, we determined a word length and a word count for each utterance spoken. Table 1 identifies the average word utterance and total number of utterances by teachers and students in the two lessons.

	Japan	US
Teacher	30.5 word average 144 total utterances 319 words – longest	12.5 word average 444 total utterances 224 words - longest
Students	4.9 word average 136 total utterances 88 words - longest	4.7 word average 569 total utterances 28 words - longest

Table 1: Average word utterance and total word utterances

The table provides an initial glimpse into the differences in utterances between the two classrooms. First, the average length of utterance by the Japan teacher is more than double that of the US teacher (30.5 to 12.5) words per utterance. Not surprising then is that the Japan teacher has only a third of the total number of utterances (144) in comparison to the US teacher (444). The student utterances in both the Japan and US classrooms are similar in that they are just less than five words on average (4.9 to 4.7), but the US students have more than four times as many utterances (569) as the Japan students (136). In addition, the utterance length ranged from 1 to 88 words for the Japan students and from 1 to 319 for the Japan teacher. The utterance length for US students ranged from 1 to 28 words and 1 to 224 words for the teacher. The number of utterances and words spoken in the TIMSS Japan and US video lessons were also reported by Kawankak and Stigler (1999). Although they used a different definition of utterance based on sentences or phrases serving a single function, the results for the number of words spoken are similar to what we found. In their work they found that on average over four lessons, 90% of the total words spoken were by the Japan teacher. Similarly, in the US classroom, 88% of total words were spoken by the US teacher.

After modifying the transcript and doing an initial analysis, we created a bar-mekko chart (i.e., a variable width column chart) for both lessons. A mekko chart is a two-dimensional stacked bar graph where both the height and width provide information about the data. In this case, the height of a bar is the utterance length in words and the width of the bar is the length of speaking time (graphed in seconds).

In the Japan lesson (Figure 4), the 319-word utterance occurs at approximately 33 minutes into class. It involves the teacher describing various methods for solving inequality problems based on solutions provided previously by the teacher and shared by students. In the US lesson (Figure 5), a 224-word utterance by the teacher occurs at approximately the four-minute mark. It is shortly after students enter the class and are seated. During the utterance, she explains that the topic they are beginning is exponents and she reminds them of their previous experiences with exponents by providing examples. In both graphs, the blue bars are associated with teacher utterances and the red bars are the length and time of student utterances.

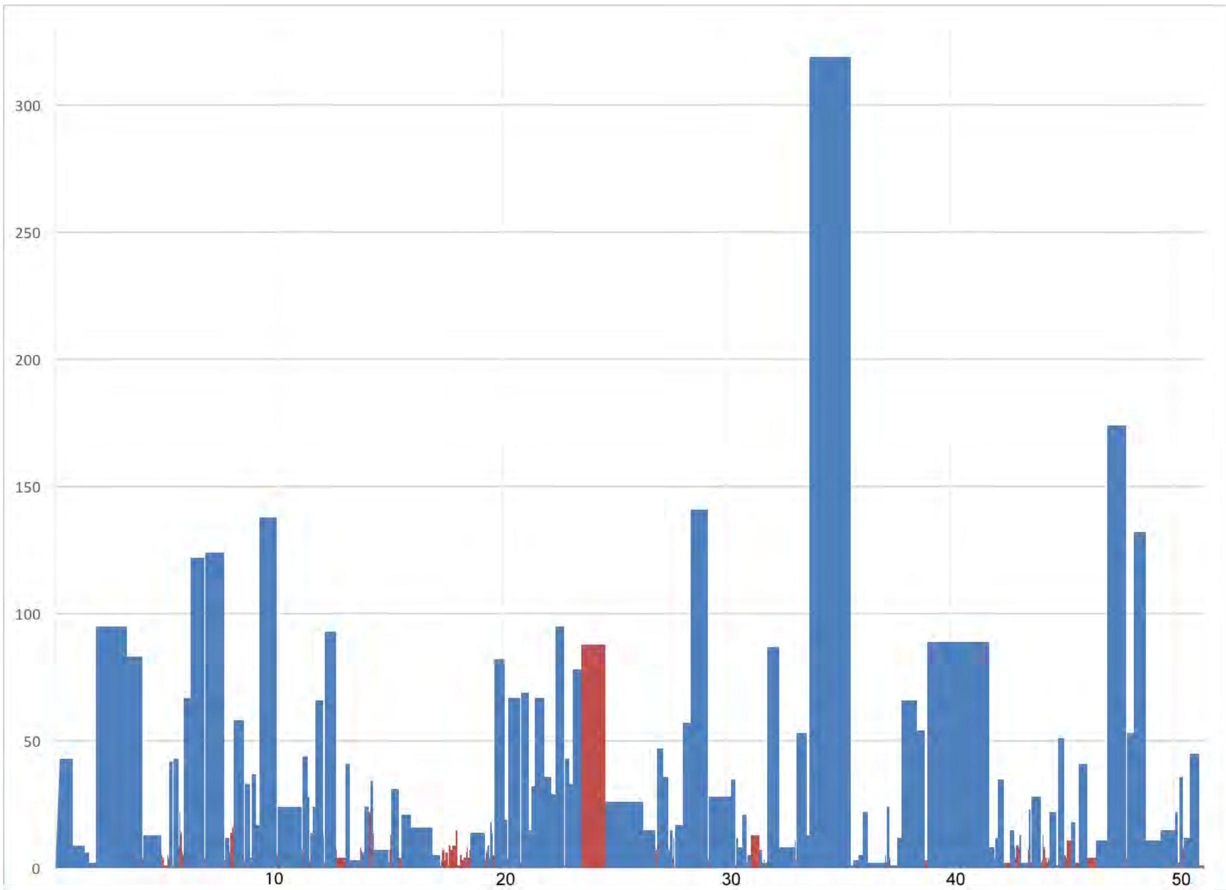

Figure 4: Japan classroom utterances by the teacher (blue) and students (red)

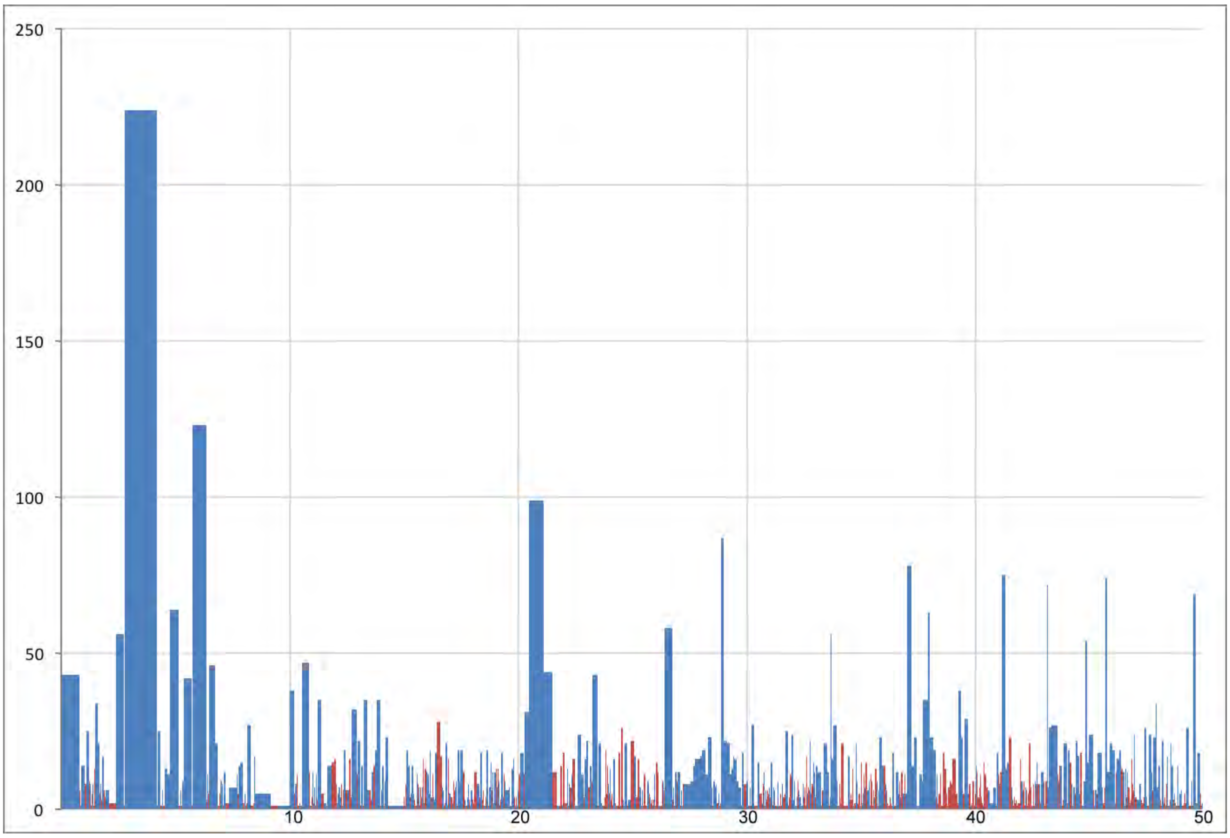

Figure 5: US classroom utterances by the teacher (blue) and students (red)

COMPARATIVE ANALYSIS

The graphs of utterances across the two lessons are visually very different. As we can see visually, the Japan teacher has multiple mid-range and long utterances throughout the lesson. The periods where the teacher is less active (starting at 17 minutes and at 43 minutes) is when students are sharing solutions with the whole class. While the teacher does have short exchanges with some students during individual seatwork, for the most part, the teacher continues to provide guidance to the whole class in the form of longer utterances. It is also important to note that while the students are less active verbally in comparison with the US lesson, the red bars do occur throughout the lesson. Students are rarely fully silent.

Comparatively, the US teacher has multiple short and mid-range utterances that occur throughout the lesson. The longer utterances tend to occur with minimal student response which are the moments when the teacher is speaking to the entire class. The frequent exchange of shorter teacher and student utterances are indicative of the teacher and student interactions during small group work. The waves of moving between large group discussion and small group work are evident in the graph. The contrast in the length and timing of utterances across the two lessons is noteworthy. Also interesting is that similar activity (e.g., instruction to the whole class) does not look the same in both classrooms. The whole class activity in the Japan lesson reveals short, but frequent student utterances throughout. By contrast, the whole class activity is often dominated by blue bars, with minimal student utterances in the US lesson.

To extend our utterance analysis somewhat further, we examined the distribution of utterance lengths across the two lessons. That is, how many one-word, two-word, three-word utterances and so on occurred by the teacher and the students. Both graphs have been put on the same scale with a maximum frequency of 100 occurrences (y-axis) and a maximum word-utterance of 50-words (x-axis). In the Japan classroom (Figure 6), this scale does not include a student utterance that was 88 words, as well as 27 teacher utterances (of which 20 utterances are between 51 and 100 words). In the US classroom (Figure 7), the scale does not include 20 one-word utterances by students (70 one-word utterances in total) and 15 teacher utterances that are more than 50-words.

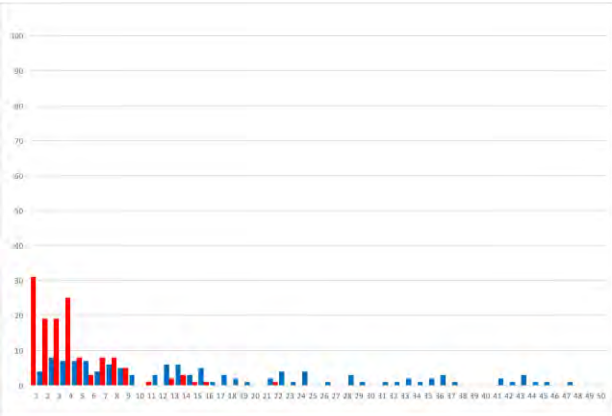

Figure 6: Japan utterance distribution

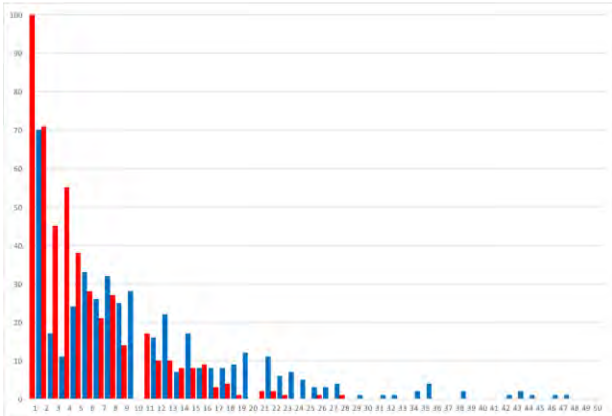

Figure 7: US utterance distribution

Overall, these figures also reveal the differences in the density of talk between the two classrooms, and where shorter utterances are the norm for students in both cases. The teacher talk in both classrooms also reveal a predominance of shorter utterances, although both have very long tails that are not captured in the images.

CONCLUDING REMARKS

Prior to beginning our analysis, we anticipated that classrooms arising from different histories, culture, and language might reveal different patterns of interaction. We found that even a simple vital sign that did not attend to word meaning, revealed quite different discourse patterns in the classrooms collectives. Although we examined the utterances between teachers and students within two lessons, we expect that the patterns revealed within one fifty-minute lesson will be self-similar to patterns revealed in other fifty-minute lessons, not only of the specific teacher, but also, for other teachers within the same school and within that school system.

We also hypothesize that by doing a comparative analysis of a larger set of lessons in different schools and school systems would allow us to recognize certain utterance distribution patterns that would provide us with insight into implicit pedagogical principles and theories of learning that are played out in the collective interactions within classrooms.

VITAL SIGN 2: (NON)ACTIONS ON THE BOARD

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THEORETICAL CONSIDERATIONS

The role of artifacts in human cognition continues to be a central focus in activity and distributed cognition theories (Susi & Ziemke, 2001). Since the early 1990s there has been a growing interest in the role of artifacts as they relate to collective behaviour in complex systems. Our focus for this vital sign is on actions (or non-actions) on the board where the board emerges as an artifact or a workspace in the environment through and which the classroom as a complex learning system coordinates actions.

Complex systems can self-organize themselves to produce organized patterns, resulting from localized neighbouring interactions within the components of the system, without any central control. As Varela (1992) notes:

> What is particularly important is that we can admit that (i) a system can have separate local components which (ii) there is no center or localized self, and yet the whole behaves as a unit and for the observer it is as if there was a coordinating agent "virtually" present at the center. (p. 11)

This complex system behaviour known as decentralized control raises the so called "coordination paradox": While in a complex system each agent appears to pursue its own agenda, somehow the collective as a whole exhibits high levels of organization or coordinated actions or behaviour (Theraulaz & Bonabeau, 1999). In 1959 a French zoologist Grassé sought to understand the mechanisms underlying decentralized control in social insects. Specifically, Grassé inquired into the so-called "coordination paradox". Grassé found out that in the coordination and regulation of termite colonies, there is the phenomenon of indirect communication mediated by modifica-tions of the environment; that is, insects interact indirectly: each insect (ants, bees, termites) affects the behaviour of other insects by indirect communication through the use of the environment, which is made of objects and artifacts such as material for the nest, or chemical traces. The environment is not a mere passive "container", but in contrast, it embeds mechanisms and processes that promote the emergence of local and global coordinated behaviours. Grassé coined this phenomenon or mechanism stigmergy. The term stigmergy is formed from the Greek words stigma "sign" and ergon "action," and captures the notion that an agent's actions leave signs in the environment, signs that it and other agents sense and that determine their subsequent actions.

Despite the differences between social insects and other animal systems (flocks of birds, schools of fish, etc.), these animal systems appear to exhibit similar collective behaviours suggesting the possibility of stigmergy as a mechanism underlying the collective coordination of actions with human systems. However, ant-like agents do not exploit the same cognitive ability as humans and the ant environment is quite different and elementary, including pheromone-like signs/signals (Ricci et al., 2007). As Maturana (2002) notes, unlike other animal systems, human beings exist in language as consensual coordinations of coordinations of behaviours.

> As we [humans] language, objects arise as aspects of our languaging with others, they do not exist by themselves. That is, objects arise in language as operations of coordinations of coordinations of doings that stand as coordinations of doings about which we recursively coordinate our doings as languaging beings. (p. 29)

Thus for human systems, the environment includes signs or signals that are subject to an interpretation in the context of a shared, conventional system of signs. Also the environment is articulated, and is typically composed of artifacts, which build up the social workspace, or field of work (Ricci et al., 2007). There are many examples of human-human stigmergy (e.g., formation of trails, document editing, social networks, etc.). As Parunak (2005) notes, "it would be more difficult to show a functioning human institution that is not stigmergic, than it is to find examples of human stigmergy" (p. 163). Parunak (2005) proposes a framework that can be used to analyze a stigmergic system. A stigmergic system comprises of a population of agents and an environment in which they are immersed. Each agent has an internal state, which generally is not directly visible to other agents; sensors that give it access to some of the environment's state variables; actuators that enable it to change some of the environment's state variables; a program (its "dynamics") that maps from its current internal state and its sensor readings to changes in its state and commands given to its sensors and actuators. The environment has a state, certain aspects of which generally are visible to the agents; a program (its "dynamics") that governs the evolution of its state over time.

We are interested in learning about a classroom as a stigmergic (complex) system comprised of students and teacher as agents and environment in which they are immersed as articulated by classroom artifacts that play a vital role in the coordination of coordination of behaviours/actions. Examples of artifacts include boards such as those that exist in mathematics classrooms around the world, flip chart papers, post-it notes, and computational environments. The board in the classroom can be used in various ways ranging from a static space for conveying information to a dynamic intellectual commons for coordinating actions. Given this, we chose to focus on (non)actions/activity on the board; that is, its state and dynamics as a vital sign of classroom life. In the following section, we describe a tool for observing the vital sign.

DESCRIPTION OF THE TOOL

The lesson storyboards in Figures 2 and 3 suggest the importance of the board to the activity in the classroom. In the Japan classroom we see students at the board writing solutions, and the teacher writing on and pointing to the board. In the US lesson we also see the teacher writing on and pointing to the board. To track the activity on the board—its state and dynamics—we identified four possible (non)actions: (1) pointing to something on the board; (2) erasing something from the board, (3) adding something on the board, and (4) something that was previously on the board but no more is added. We created the following four colour codes to identify the possible (non)actions on the board (see Figure 8).

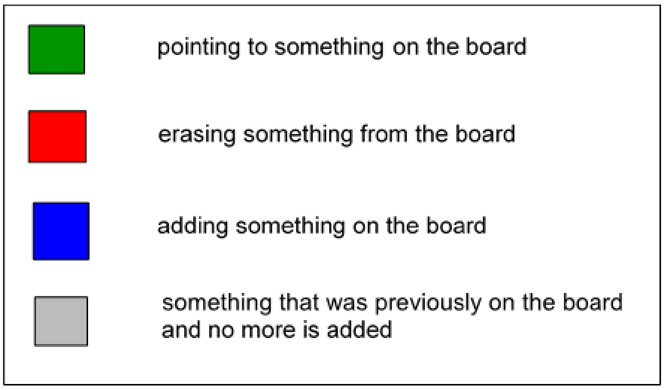

Figure 8: Colour codes for board (non)actions

Our analysis used the Japan and US videos as the sole source of data. We coded every 15 seconds of video to identify one or more of the activities shown in Figure 8. That is, if the teacher wrote on the board, pointed to it, and then erased the board, then three activities (blue, green, and red) was recorded during the full 15 second time period. Some interpretation was needed if the camera was not focused on the board, but activity was occurring based on the audio of the lesson. Figures 9 and 10 offer the visual vital signs of the board (non)actions for the Japan and US respectively over the two 50-minute lessons.

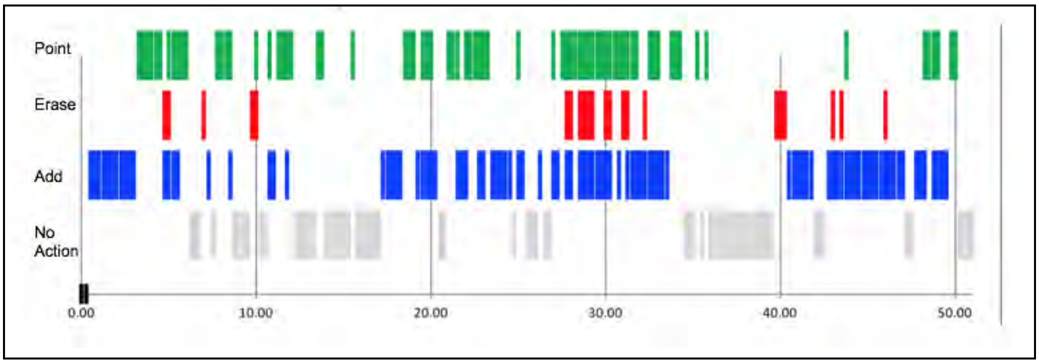

Figure 9: Japan lesson – board (non)actions

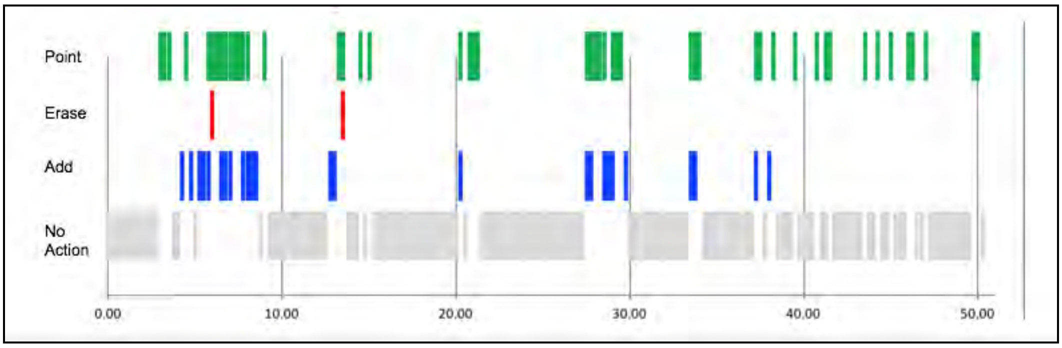

Figure 10: US lesson – board (non)actions

RESULTS AND DISCUSSION

Given the vital sign, certain patterns can be observed from the board (non)actions of the US and Japan lessons. In the Japan lesson, most of the actions on the board consisted of adding (new content) on the board and pointing to the board. There were also noticeable (non)actions for the actions of erasing and adding nothing to something that was previously there. Further still, at the beginning of the lesson—it seems as though there was nothing on the board and new content was added for about 2 minutes and at the end of the observation time there is a 1-minute period during which no actions were performed on the board yet something remained on the board.

In the US lesson, most of the actions on the board consisted of adding nothing to something that was previously there and pointing to something on the board. There was very little erasing on the board with only two instances. There were noticeable (non)actions for the action of adding (new content) to the board. At the beginning of the lesson, it appeared as if there was content which was previously on the board and nothing was added—with no other actions for about 2.5 minutes. The next action that followed this was pointing to the board.

Comparison between board (non)actions for the Japan and US lessons revealed differences between the dynamics of board (non)actions. What (non)action(s) observed as predominant in the Japan lesson were the opposite of the US lesson. In the Japan lesson the predominant actions were adding (new content) on the board and pointing to the board while in the US lesson most of the actions consisted of adding nothing to the content/things that were previously on the board and pointing to the board. A remarkable difference between the dynamics of the Japan and US lesson board (non)activity concerns the action of erasing. There was very little erasing from the board in the US lesson compared to what occurred in the Japan lesson. The content on the board in the US lesson appeared to be static, while the content on the board in Japan lesson appeared to be dynamic. There also appeared to be more density or intensity and diversity of board (non)actions on the board in the Japan lesson than on the board in the US lesson.

In our research, we use the word collective rather than collaborative (Martin, Towers & Pirie, 2006) as we realize that not all collaborative classroom interactions and actions lead to the emergence of a collective learning system. The differences in the dynamics concerning the board (non)actions for the Japan and US lessons gives rise to a question regarding how such observation might provide insight into the complexity of the class as a collective learning system.

The dynamics of a stigmergic (complex) system both for agents and their environment are typically nonlinear, and their interactions are often nonlinear as well (Parunak, 2005). The dynamics enable the systems' self-organization, since they allow the system to explore its state space efficiently. We wonder how the dynamics of the board (non)actions can inform us on the nonlinearity of classroom interactions and the potential for self-organization. In his work on building thinking mathematics classrooms, Liljedahl (2016) explored the potential for alternative work surfaces, such as poster board, flipchart paper attached to the walls, smaller whiteboards laying on desks, and vertical white boards in supporting students' thinking (thinking classroom). He observed that:

> Groups are more eager to start, there is more discussion, participation, persistence, and no-linearity when they work on the whiteboards. However, there are nuances that deserve further attention. First, although there is no significant difference in the time it takes for the groups to start discussing the problem, there are a big difference between whiteboards and flipchart paper in the time it takes before groups make their first mathematical notation. This is equally true whether groups are standing or sitting. This can be attributed to the non-permanent nature of the whiteboards. With the ease of erasing available to them students risk more and risk sooner. (p. 371)

We wonder if the difference between the board (non)actions in the Japan lesson and US lesson in terms of the action of erasing and diversity of the actions (as observed earlier) might inform us about whether or not one of these shows nonlinearity and therefore has potential for self-organization.

VITAL SIGN 3: EMERGENT IDEATIONAL NETWORK

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Université du Québec à Montréal, Canada

THE RATIONALE

In Mindstorms, Papert (1980) insisted on the enrichment of the mathematical environment in which children are put into, hence expressing that the richer the mathematical ideas in a classroom, the richer the learning environment for students. Another related aspect is how these ideas emerge in a classroom and follow one another to produce this rich mathematical environment for the classroom. Taking these ideas seriously, and aiming to explore them in depth, we are developing mechanisms to model the dynamic unfolding of mathematical ideas in a classroom, which are making up the mathematical environment in which the entire classroom is plunged into.

This modelling technique is done through building up networks of ideas, which illustrate chronologically how one mathematical idea gets dynamically put forward one after the other, and are (possibly) connected during a classroom episode; creating by the same token the mathematical environment of that classroom. The suspicion is that the density of ideas emerging in a classroom and their strong interconnectedness can play an important role in the richness of the mathematical classroom environment.

THE TECHNIQUE

In this research forum, this modelling technique is illustrated from the analysis of classroom episodes. We analyzed the Japan and US TIMMS video lessons using the transcripts and videos available. Technically speaking, every episode starts from a task, a theme or an explanation given in the classroom. From this, other ideas then emerge and unfold in the classroom: we aim at modelling these through a network. To conduct this modelling, each mathematical idea emerging during an episode is represented in a bubble, connected through links to show after what other idea this former idea emerged. If an idea is not connected to another one, then no link is traced. As the same idea is being forwarded anew in the discussion, its bubble gets re-underlined. And, if this idea is related anew to the same idea from which it emerged, its linked connection gets enlarged, resulting in some ideas being more strongly related to the other in the unfolding of the episode.

During an episode, when a question is asked, be it from the teacher or the student, a large red question mark is placed beside the idea that provoked that question. From there the modelling continues, with ideas shared emerging from that question. When the exploration of that question ends, and that the new ideas shared are not linked to it anymore, the thread of linked bubbles can possibly, if relevant, return to previous ideas or tasks under investigation.

MODELLING THE EPISODES

As described in the introductory paper to this research forum, the US 8th grade class-room concerns operations with exponents (e.g., $a^m \cdot a^n = a^{m+n}$), and the Japan 8th grade classroom concerns the modelling of expressions with inequalities, each lasting about 50 minutes.

Before entering the details of each idea emerging one after the other in the episode, a look at the general network structure in terms of its "moments" (1st level) appears interesting. In effect, although each episode appears quite different at first glance when watching the videos, the modelling of each renders similar networks. For this 1st level of structure, the main "moments" of the episodes, a similar pattern appears in each network (with an additional break in the US episode): each episode starts with an introduction on the ideas of the day (exponents, inequalities), leading to the exploration of a task, itself leading to the discovery of a variety of rules of exponents and modelling with a system of inequalities. Figure 11 shows the general structure of the modelling, the 1st level, with general "moments" placed in solid boxes.

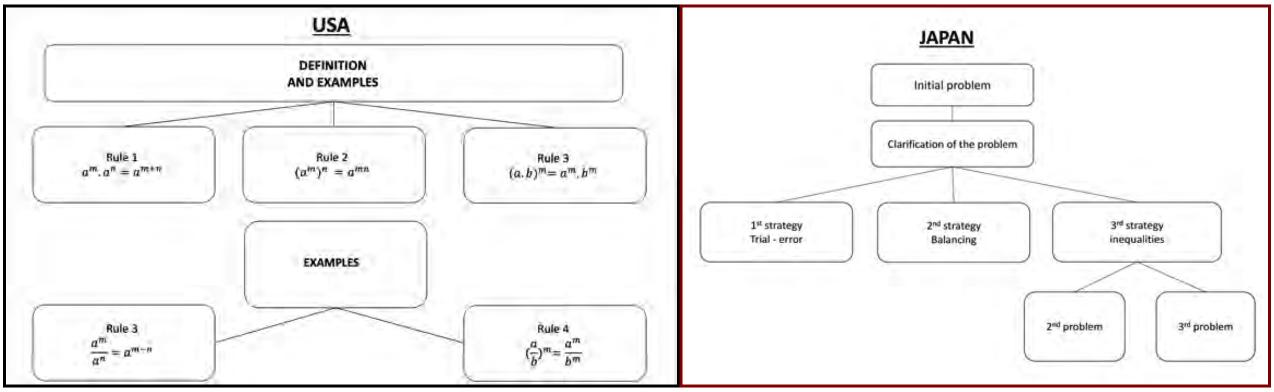

Figure 11: 1st level of networks for US and Japan lessons

However, because the modelling aims at delving into precise details at the level of ideas, other levels are needed. A 2nd level of networks reveals the "themes" addressed for each moment, tracing links between the specific "themes" inside the "moments", offering a more precise network of ideas emerging. Again, as shown in Figure 12, both episodes offer similar networks.

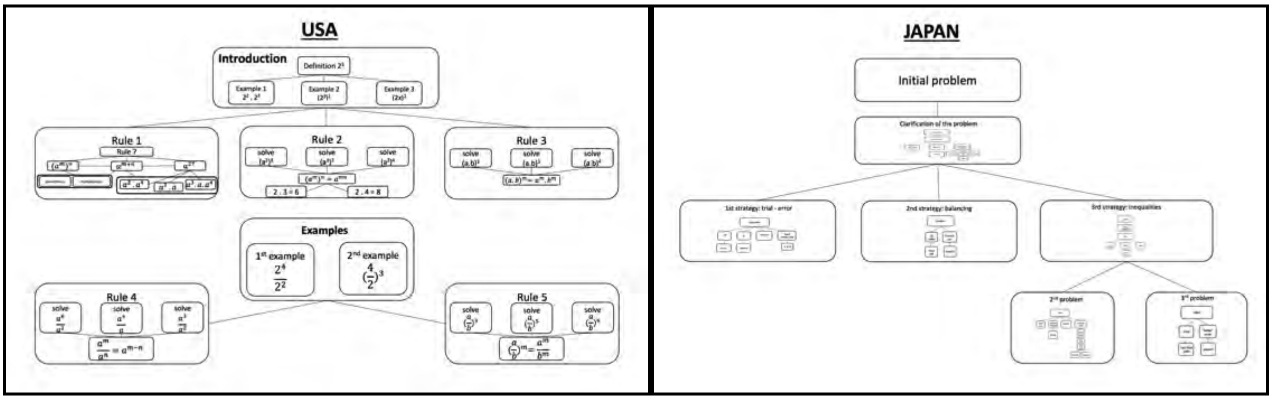

Figure 12: 2nd level of networks of ideas for US and Japan lessons

One particular aspect worth noting in both US and Japan networks is that more or less the same structure re-appears in a finer breakdown of the ideas. There appears to be a fractal-like structure present in the networks; that is, when the details of the larger network are examined closely, a smaller network of the same structure is revealed. Our current analysis of a 3rd level of detail, focused on precise ideas addressed within a "theme", also illustrates the same sort of geometrical fractal-like patterning appearing in the larger themes and structure (to be presented at the Research Forum). One other element worth highlighting is that for the most part no idea appears to be asserted more than once, at least not in an explicit manner, as the classroom goes forward and solves the problems or finds the rules needed and builds toward other mathematical ideas. (This explains why the links between themes or ideas are single width.) The relevance and significance of this finding needs to be investigated further through subsequent modelling of classroom episodes.

REFINING THE MODELLING

Whereas the modelling offers a "structure" of the network of moments, themes, and ideas, precise details are lacking. One of these details concerns the teacher/student ratio in terms of provenance of ideas. When using a colour-coded scheme (red for teacher and yellow for students), both modelling illustrate qualitative similarities with an important majority of ideas offered in the mathematical environment of the classroom coming from the teacher (which relates directly to the Utterance Length and Distribution Vital Sign).

Another one of these details concerns the content, in order to gain insight on the nature of the mathematical ideas shared. Skemp's (1978) well-known and well-tested distinction between instrumental and relational understanding, that he extended to mathematics itself, was used as a first attempt to categorize the nature of the ideas/explanations given (green for instrumental and blue for relational). This, however, did not map all the ideas shared, since some ideas were only answers given to a question with no explanations or facts directly asserted. Those were thus coloured in pale green. The modelling outlines again very similar networks of colours for both US and Japan, with all ideas being placed in the green (that is, no ideas representing relational mathematics were found in the ideas shared). These results were surprising.

As well, the comparison of these networks produced for both US and Japan episodes with other previously conducted networks from problem-solving sessions (see McGarvey et al., 2017), highlighted important differences. Consider one of these networks, illustrated in Figure 13, coming from the exploration of a ratio problem of mixture of water and juice (see Noelting, 1982).

First, although the fractal-like shape is again apparent for this network, the level of interconnectedness between ideas appears denser: numerous links being traced between the ideas, no mathematical ideas hanging as isolated bits of information left aside in the network, frequent repetition of the same ideas (thickening the links between these ideas and others). In short, all ideas appear to be strongly interconnected.

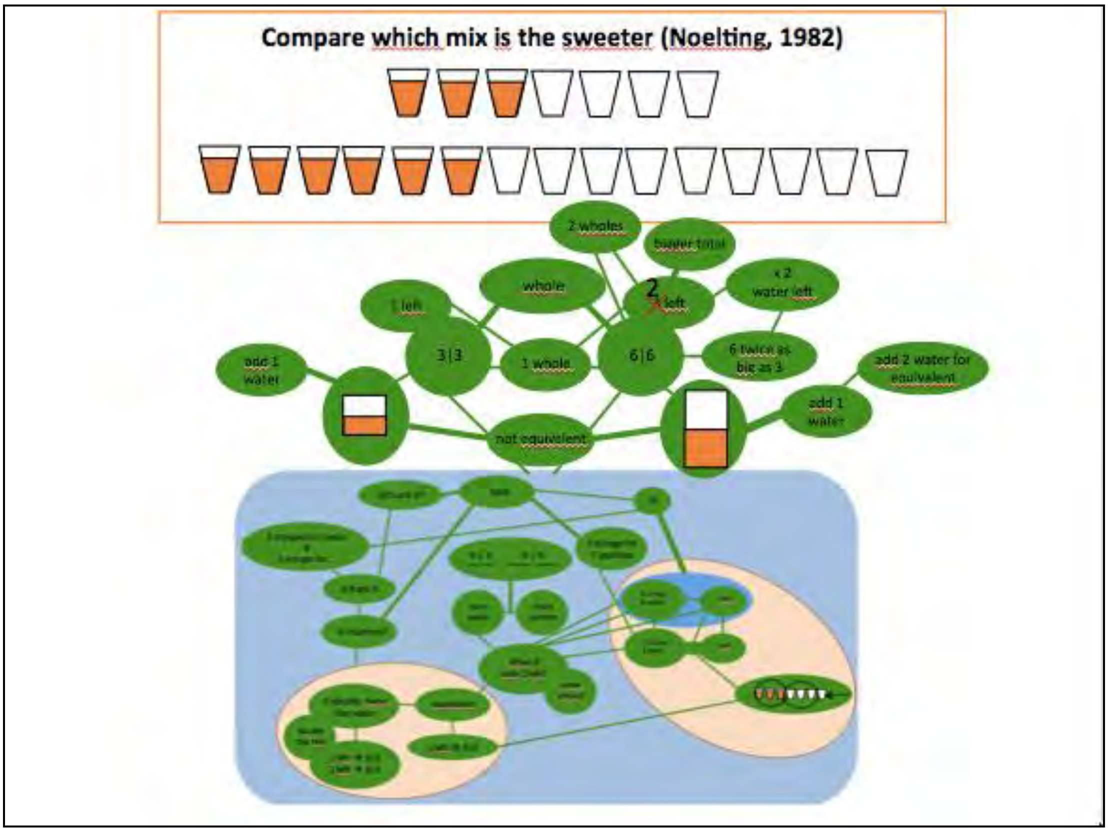

Figure 13: Final ideational network from a problem-solving session

Second, the teacher/student ratio in terms of provenance of ideas is very different, with as much if not more of the ideas coming from students. *Third*, the questions also frequently came from students themselves, which then reoriented the exploration of the episode for the entire group (see also Cobb et al., 1994). *Fourth*, the nature of the mathematics, when using the answer/fact, instrumental, relational distinctions is quite different with a majority of the ideas shared being in the relational mathematics category; this is a significant difference between the episodes, independent of the differences in the network.

Finally, one cannot work at modelling networks of ideas without considering the nature of the networks themselves. By referring to network theory (e.g., Barabasi, 2003), and the image offered in Figure 14, the Japan and US classroom episodes appear to be more aligned with a centralized network, where most ideas have their origins in a specific node and follows from it linearly. Put bluntly, these networks have important weaknesses as ideas are not robustly built upon a variety of sources of ideas, since all mostly come out of one central idea and follow from it: if that central idea disappears, the network collapses.

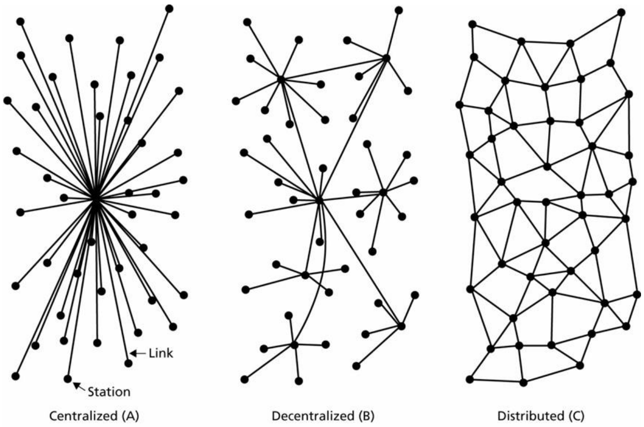

Figure 14: Types of network architecture (Baran, 1964, p. 2)

In contrast, our previous modelling appears more like a decentralized network, with important nodes but also strong interconnectedness between ideas and new ideas emerging from others and culminating at times toward an additional new but reduced-format network on its own; ideas bundle up on top of others and connect to them. The robustness of this sort of network is much stronger and, in contrast to the centralized network, the loss of one idea can be recuperated from others. (Still, the distributed network would be more robust. We have *not yet* modelled classroom episodes with this sort of network, although it could be argued that the modelling about the Noelting problem is close to it with its numerous interconnections.)

FINAL REMARKS ON THE MODELLING

The comparison of the varied networks (shared at the Research Forum) are of significance in understanding the richness of the mathematical environment of the classroom in which the collective is immersed (Papert, 1980). The distinctions highlighted and the modelling make salient a variety of differences between the episodes as well as similarities. Through these modelling techniques, the comparison of a variety of classrooms offers a sense of the sort of similarities and of differences from one classroom to the next in terms of the emergence of ideas and the richness of the mathematical environment they provide to the collective. In the long-run, this modelling aims to offer a better understanding of the structure of classroom episodes, especially those that appear rich mathematically speaking. Hence, the modelling aims to go beyond our usual techniques of reporting on mathematical events happening in a classroom that aims to highlight its richness or opportunities for advancing mathematical understandings. Although they are insightful and have been used for a while in mathematics education research to offer a glimpse at classroom events, the modelling proposed here attempts to offer an understanding of how a classroom functions structurally speaking in terms of the emergence of its mathematical ideas. The detachment from the individual in profit of the collective enables this shift in reporting and analyzing classroom episodes.

Obviously, at this stage of the research, these modelling techniques are still in development. The research team constantly attempts to improve the modelling to obtain networks that illustrate classroom episodes *and,* at the same time, the modelling allows us to view dimensions that were not salient from only watching the videotapes of these episodes. To some extent, the work conducted on modelling is similar to tinkering techniques seen, for example, in biology (see Watson, 1980): the modelling attempts to capture features of the episodes on the basis of what one sees in it, and at the same time offers to view unseen aspects of the episode, which recursively leads to adjustments of the model(ling) to attempt to see more while at the same time continuing being truthful to the episode itself. The interplay of *seen*, *unseen* and *unforeseen* aspects of the episode makes this modelling process dynamic and enriching for the data under scrutiny.

VITAL SIGN 4: PIRIE-KIEREN MODEL AND THE DYNAMICS OF IDEAS

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[1]University of Alberta, [2]University of Victoria

BACKGROUND

In 1989 Pirie and Kieren introduced a model and a theory for the growth of mathematical understanding. The authors characterized mathematical understanding as an embodied process that was inherently dynamic, levelled but non-linear and recursive (Pirie & Kieren, 1994). The model featured eight nested levels: *Primitive Knowing* as "the starting place for the growth of any particular mathematical understanding" (p. 170); *Image Making* as the activity by which to "make distinctions in previous knowing and use it in new ways" (p. 170); *Image Having* as the "use [of] a mental construct about a topic without having to do the particular activities which brought it about" (p. 170); *Property Noticing* as the action by which to "manipulate or combine aspects of images to construct context specific, relevant properties" (p. 170); *Formalising* as activity which "abstracts a method or common quality from the previous image dependent know how which characterised noticed properties" (p. 170); *Observing* as "reflect[ing] on and coordinat[ing] formal activity and express[ing] coordinations as theorems" (p. 171); *Structuring* as involving "formal observations as a theory" (p. 171); and *Inventising* which entails the "break[ing] away from preconceptions ... and creat[ing] new questions [that] might grow into a totally new concept" (p. 171). The nested structure of the model reflects each level as including all inner levels as well as being integral to all outer levels. To date, the theory has almost exclusively been used to illuminate the understanding of individual students. In contrast, we use Pirie and Kieren's theory to attend to the emergence and dynamics of ideas at the collective level, in mathematics classes, as suggested by Thom and Glanfield (in press); Kieren and Simmt (2002); Martin and Towers (2003; 2015); Davis and Simmt

(2003); and Pirie and Kieren (1994). For the Japan and US lessons, we identified the level at which the ideas emerged, monitored the ideas as they were (re)iterated or elaborated upon, and tracked the ideas during each lesson as they moved back and forth across the different levels of the model.

ANALYSIS

The first stage of our analysis was to identify concepts and ideas within the lessons. There were two ideas in the Japan lesson around the concept of inequality. Idea 1 (I1[JP]) involved the procedure(s) used to solve an inequality and Idea 2 (I2[JP]) related to how an inequality expression could be used to model a specific context. Five ideas were observed in the US lesson connected to the concept of exponents. These were: composition of powers (I1[US]); exponential growth (I2[US]); multiplying powers with the same bases (I3[US]); dividing powers with the same bases (I4[US]); and multiplying powers with different bases (I5[US]). We observed the emergence, (re)iteration(s), and elaboration(s) of mathematical ideas involving the identified concept(s) while using the Pirie-Kieren theory to code the ideas within the lessons. Finally, we mapped the emergence and dynamics of the ideas on the Pirie-Kieren model (see Figures 15-20).

RESULTS

In this section, we illustrate the emergence, (re)iteration, and elaboration of the ideas during the two lessons as they relate to the Pirie-Kieren model.

Japan Lesson

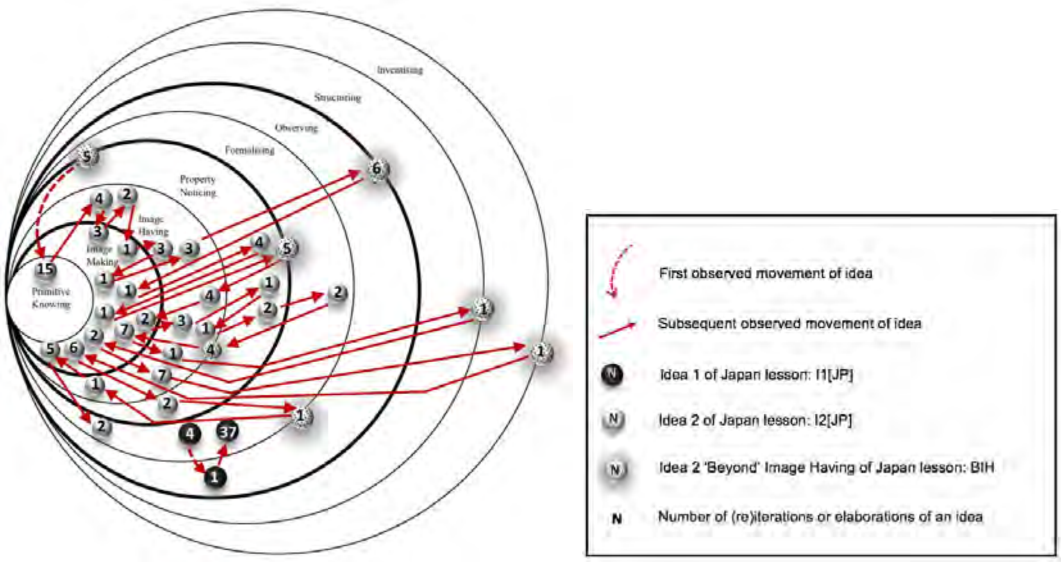

Figure 15: 0:00-16:59 of the Japan lesson

The first 17 minutes of the lesson are represented in Figure 15. Five minutes 33 seconds of this period were not coded. Two minutes 57 seconds consisted of going over homework related to I1[JP]. I1[JP] emerged, and for the most part, stayed at the Formalising level. The balance of time, 8 minutes 30 seconds, was spent on I2[JP]. Interestingly, I2[JP] arose in manners that were not specific to any one level in the model but indeed, clearly beyond Image Having. To distinguish these events, we mapped the moments in which I2[JP] occurred Beyond Image Having as dotted spheres on the boundaries between levels. In addition to this, and unlike I1[JP], I2[JP] moved across levels, back and forth, from Primitive Knowing through to Formalising, and Beyond Image Having.

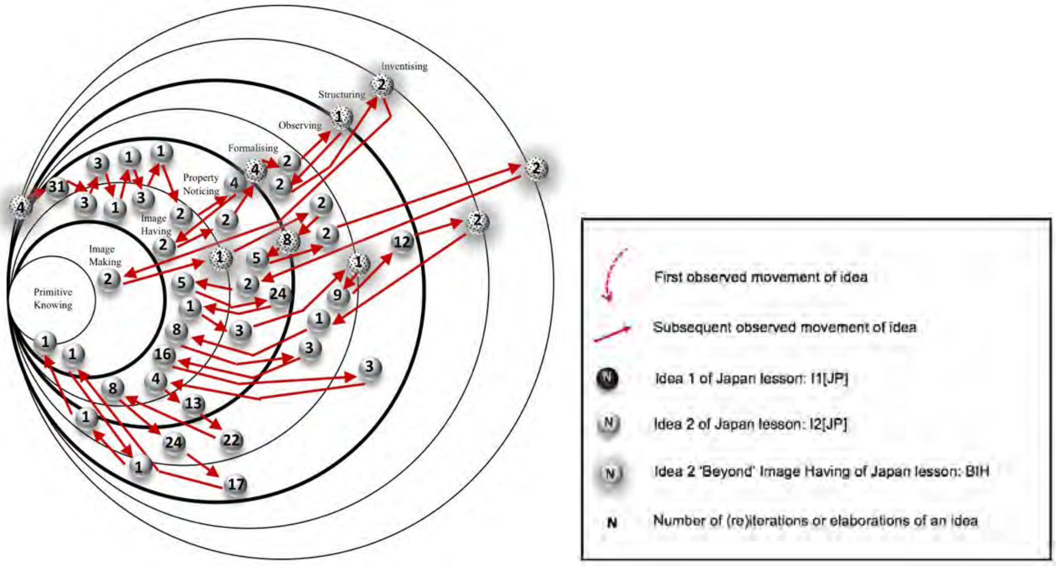

Figure 16: 17:00 to 34:59 of the Japan lesson

Figure 16 represents the next 18 minutes of the Japan Lesson. Except for 1 minute 38 seconds that were not coded, the rest of the period involved I2[JP]. I2[JP] moved back and forth through the levels of Image Making to Observing, and Beyond Image Having. It is interesting to note that most of the elaborations and (re)iterations happened between and among Property Noticing, Formalizing, and Observing.

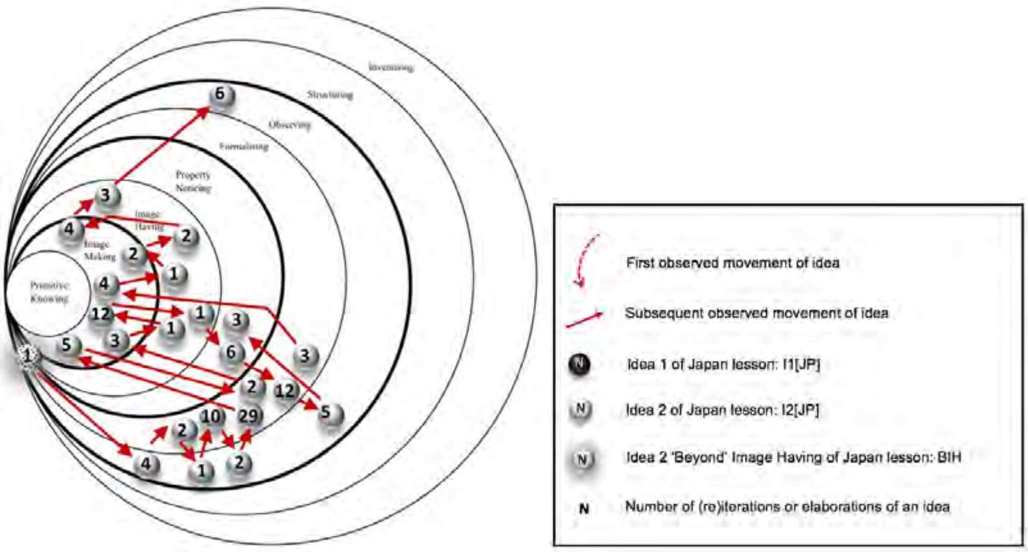

Figure 17: 35:00 to 51:36 of the Japan lesson

Figure 17 illustrates the final 16 minutes 36 seconds of the Japan Lesson. In this time period, 12 minutes 16 seconds were coded and the focus of the collective remained on I2[JP]. We observed I2[JP] moving back and forth through *Image Making* to *Observing*, and *Beyond Image Having*. Most of the elaborations and (re)iterations took place within and across *Image Making*, *Formalising*, and *Observing* with fewer instances in *Image Having* and *Property Noticing*. While both Figures 16 and 17 show I2[JP] as primarily in *Property Noticing*, *Formalising*, and *Observing*, there are many more instances in which I2[JP] occurred in Image Having in the second part of the lesson (i.e., Figure 16) than the third (i.e., Figure 17).

US Lesson

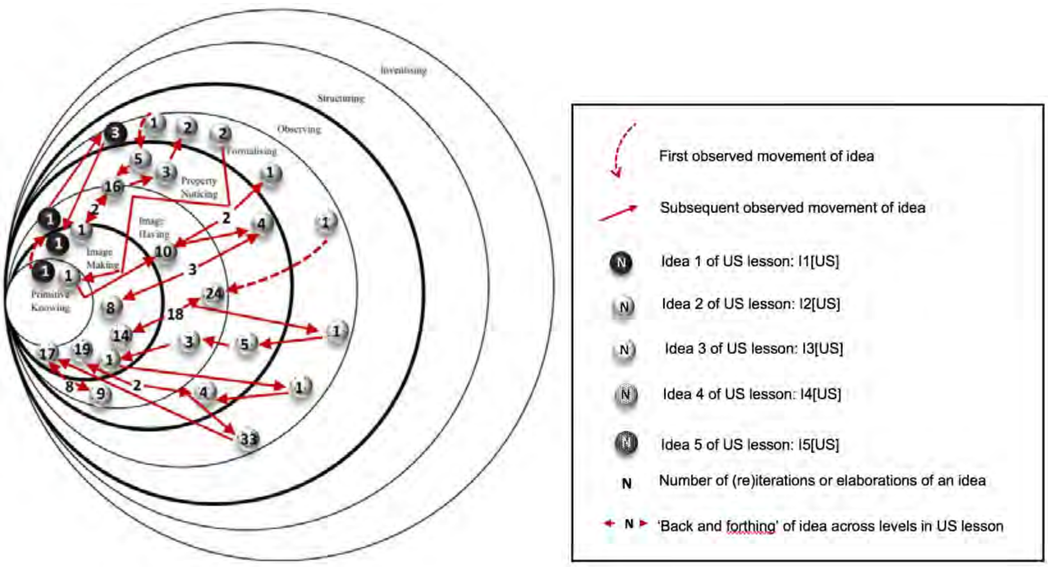

Figure 18: 0:00-16:59 of US lesson

Figure 18 illustrates the first 17 minutes of the US Lesson. In this first part of the lesson, 4 minutes 53 seconds were not coded. I1[US], I2[US], and I3[US] all occurred during this period. I1[US] was only present for about 40 seconds. I2[US] was evident in the collective for 2 minutes 15 seconds and moved across *Primitive Knowing*, *Image Making*, *Image Having*, and *Formalising* levels. I3[US] was observed at the *Formalising* level and then moved between *Image Having*, *Image Making*, and *Property Noticing*. What we found intriguing about this part of the lesson was the number of 'back and forths' of I3[US] between *Image Having* and *Image Making*.

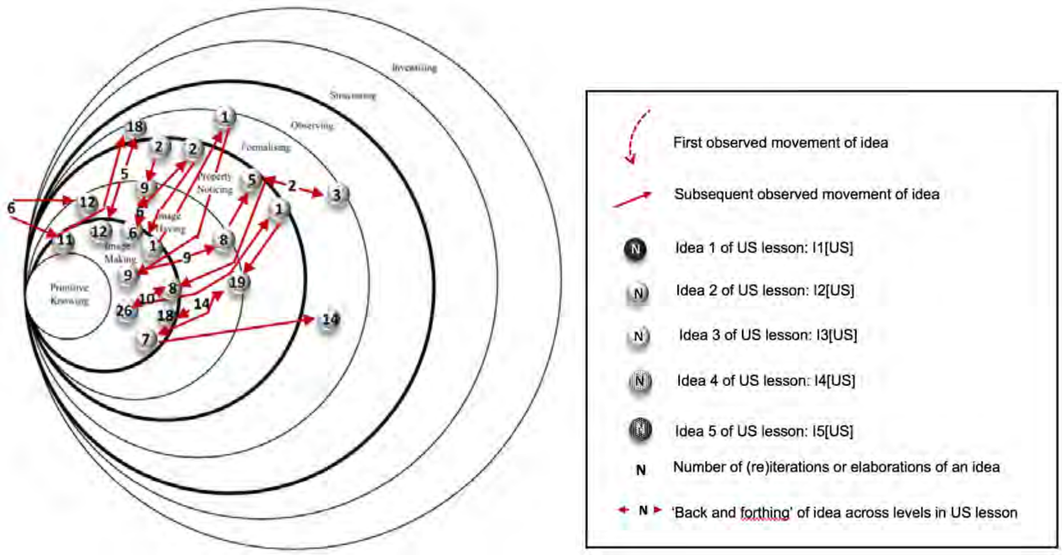

Figure 19a: 17:00 - 21:44 of the US lesson

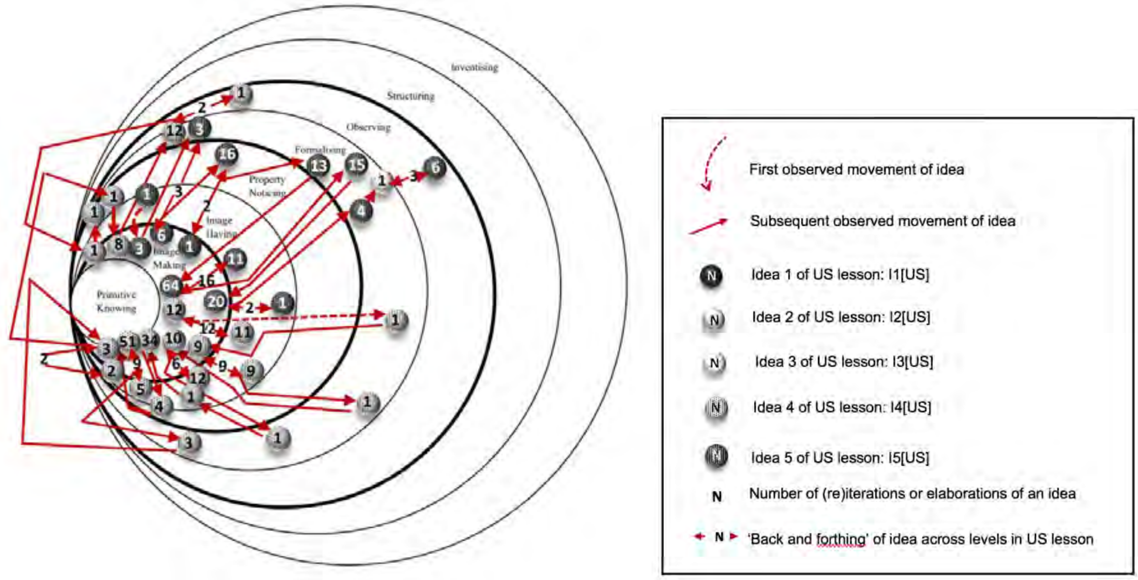

Figure 19b: 21:45 - 34:59 of the US lesson

Figures 19a & 19b represents the next 18 minutes of the US Lesson. Only 1 minute 28 seconds was not coded in this time frame. Figure 19a illustrates the dynamics of I3[US]. Figure 19b shows the emergence and dynamics of I4[US] and I5[US] as well as the single elaboration of I3[US]. Remarkably, there were so many observed elaborations and (re)iterations of the I4[US] and I5[US] in *Image Making* during this period, we required two models in order to map them. There were also several instances where ideas moved back and forth between *Image Making* and other levels. For example, in Figure 19b, I5[US] can be seen as moving back and forth between *Image Making* and *Image Having* 16 times. In these moments, I5[US] was reiterated or elaborated upon 64 times in *Image Making* and 11 times in *Image Having*.

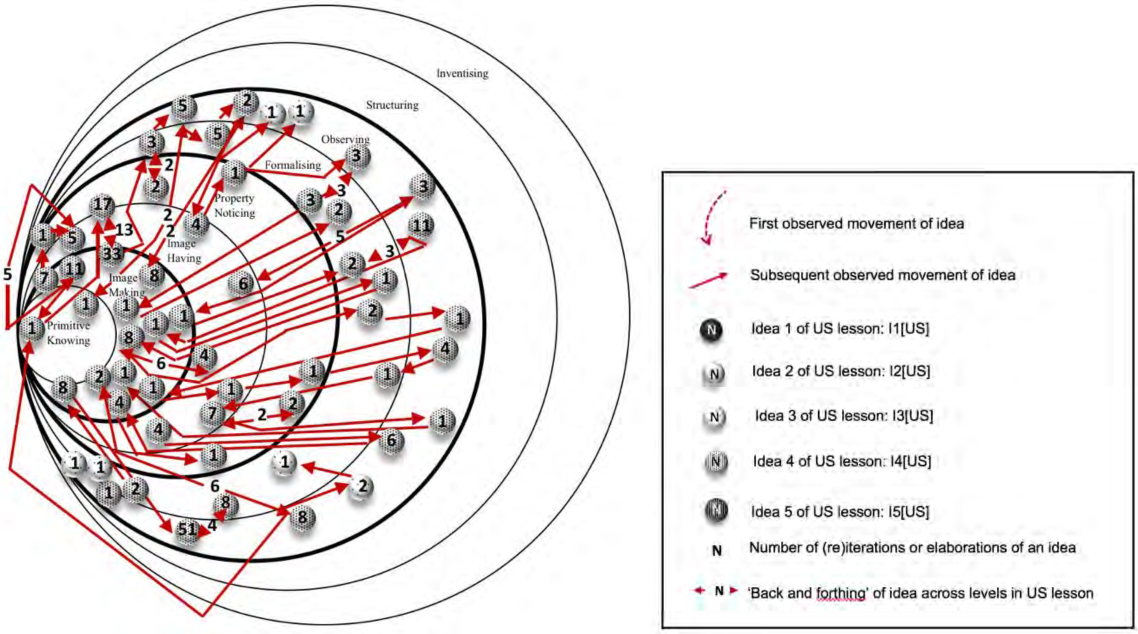

Figure 20: 35:00 to 50:18 of the US lesson

Figure 20 accounts for the final 15 minutes 18 seconds of the US lesson. In the time period, 13 minutes 6 seconds were coded. We observed 7 instances in which I3[US] was elaborated or reiterated. I4[US] can be seen as moving back and forth through *Primitive Knowing* to *Observing*. Most of the elaborations and (re)iterations of I4[US] took place within *Image Making*; similarly, to Figure 19b. However, we noticed more events involving the elaboration and reiteration of ideas within the *Observing* level than previously, as shown in Figures 18, 19a, or 19b.

DESCRIPTION AND DISCUSSION OF THE RESULTS

Our reflection on the analysis of the two lessons has enabled three new observations. These relate to *Beyond Image Having* in the Japan lesson, complementary qualities within levels of the Pirie-Kieren model during the US lesson, and the absence of any *Structuring* or *Inventising* in either of the lessons.

We described earlier, several instances during the Japan lesson where I2[JP] emerged *Beyond Image Having* but was not specific to any one level in the Pirie-Kieren model. We did not find such cases in the US lesson. Moreover, in Figures 15-17, there are episodes in which I2[JP] moved between *Image Making* and *Observing* and *Beyond Image Having* several times. Then, near the end of the lesson, as illustrated in Figure 17, only one instance of I2[JP] *Beyond Image Having* was observed and predominantly, I2[JP] was elaborated upon and reiterated at the *Formalising* level. Given this, we conjecture that the decreased activity *Beyond Image Having* and the increased activity within the *Formalising* level might suggest a consensual coordination of coordination (Maturana, 2002) of I2[JP] within the collective.

Pirie and Kieren (1994) theorised that each level beyond PK involved the "complementarity of acting and expressing" (p. 175). The authors suggested that growth "occurs through, at least, first acting then expressing, but more often through to-and-fro movement between these complementary aspects" (p. 175) and chose "six verbs, doing, and reviewing seeing and saying, predicting and recording, as labels for the acting/expressing complementarities within the image making, image having, and property noticing" (p. 175) levels. Particularly in the US Lesson, we noticed marked differences within the *Image Making*, *Image Having*, and *Property Noticing* levels of the Pirie-Kieren model. Consequently, we suggest these differences might be exemplars of the complementary aspects regarding the model levels which Pirie and Kieren described.

Lastly, while we observed no instances of *Structuring* or *Inventising* during the Japan or the US lesson, we surmise this may be due to the time space in which our examination was framed. That is, there may not have been sufficient time to see these events occur. Perhaps if the study continued over a series of classes, we might observe the extension of ideas into these levels of Pirie-Kieren model.

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WORKING GROUPS

INTEGRATING MATHEMATICS IN STEM EDUCATION: AN INTERNATIONAL PERSPECTIVE

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Since there has been limited attention to STEM education research in the mathematics education community (English, 2016), last year we convened the first Discussion Group on STEM education at the PME conference in Singapore (Anderson &Li, 2017). With over 35 participants from more than 12 countries, it was evident there was a need for a community of scholars from the mathematics education research community to consider the role of mathematics in STEM education in schools, to critique the approaches to integrating mathematics with the other disciplines in STEM education, and to share the challenges of coordinating competing and dissimilar 'practices' across the diverse disciplines in STEM (Hobbs, Cripps Clark, & Plant, 2018). Participants were keen to continue the conversation this year and to consider possible contributions to a scholarly publication on STEM education. The goal of our working group sessions is thus to provide the opportunity for mathematics educators and researchers from diverse contexts to connect, share experiences and develop chapter proposals for the book. The objective of the Springer volume *Integrated Approaches to STEM Education: An international perspective* is to provide an international audit of current research and practice with recommendations for researchers, policy makers and teachers about integrated STEM education. With open invitation for contributions from diverse professional communities including PME, the book will evaluate the efficacy of integrated STEM education as it is currently practiced and provide a platform for further research and collaborations in the STEM education research community internationally.

This working group meeting will provide a platform for international scholars to share evidence for effective practices in integrated STEM education and contribute to the theoretical and practical knowledge gained from the diversity of approaches. Many publications on STEM education focus on one or two of the separate STEM disciplines without considering the potential for delivering STEM curriculum as an integrated approach (Bybee, 2013) – this working group meeting and the subsequent publication seek to debate the efficacy of integrated STEM curriculum and instruction, providing evidence to examine and support various integrations. Our discussions will focus on the problems seen by teachers and academics working in the fields of science, technology, engineering and mathematics and the challenges in providing a set of valued practices which have demonstrated their use and viability to improve the quality of integrated STEM education. The working group discussions will consider the themes of developing a conceptual basis for integrated STEM education and the role of mathematics, reviewing historical developments in integrated STEM education policy

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 187-188). Umeå, Sweden: PME.

and practices, describing the outcomes of effective integrated STEM education approaches including curriculum design and pedagogical practices, and providing recommendations for ways forward for research and practice.

PLAN FOR WORKING GROUP SESSIONS 1 AND 2

30 mins	Session 1 – Brief introduction – outcomes from last year's discussion group and goals of this working group. Judy Anderson and Yeping Li
45 mins	Forming theme-based groups to share STEM education perspectives, approaches, and research agendas in participants' countries and education contexts with possible focuses on the role of mathematics in STEM and the ways of integrating mathematics with the other STEM subjects
15 mins	Summary of common (existing and/or emerging) themes, topic areas, questions and making connections with scholars with similar interests
15 mins	Session 2 – Summary of Session 1. Further sharing of STEM education perspectives, approaches, and research agendas in participants' countries and education contexts with a focus on mathematics in STEM.
45 mins	Discussing possible structures of chapter proposals and drafting chapter proposals
15 mins	Sharing ideas and possible structure outline for the volume
15 mins	Confirming the timeline for chapter proposals (due 15 August 2018) and the process for selecting chapters. Researchers and practitioners are invited to submit their chapter proposals, clearly describing the topic of the chapter in 250-500 words, and listing the tentative sections alongside a rough estimate of their lengths. Proposals should also include authors' information (names, affiliations, email addresses, and short bios).

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TEACHER TENSIONS AS A LENS TO UNDERSTAND TEACHERS' RESISTANCE TO CHANGE

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Teaching in general, and teaching mathematics in particular, is a complex endeavour. Looking at the teaching of mathematics through the lens of tensions allows us to represent this complexity and to capture the nuances and problems of school life (Berlak & Berlak, 1981). Within this frame, tensions can be thought of as the force that exists between two competing and equally worthwhile aims (Ball, 1993), where choosing either over the other will not produce a perfect solution (Katz & Rath, 1992). Research on mathematics teachers' tensions has revealed that tensions are endemic to teachers' practice and can serve as powerful explanatory and analytic tools for teachers' practice (Adler, 2001). Tensions assist teachers in identifying, recognising, talking about, and acting on the dilemmas they face in their practice.

From a theoretical perspective, tensions in general and teacher tensions in particular intersects with several research topics in Mathematics Education, for example professional development. Our goal with this Working Group is to jointly explore our experiences with these intersecting topics and to broaden our understanding of the role of tensions in research related to mathematics teacher practice and knowledge. Participants will be invited to bring both specialized knowledge in theoretical perspectives that link with teacher tensions and examples (fictitious or real) of relevant tensions.

The first 90-minute slot (Slot 1) will be dedicated to examining how teachers' tensions emerge in literature, as well as to extend it through our own experiences with teacher tensions (see e.g. Liljedahl et al., 2015). We will begin by presenting ways in which literature has positioned the theory of mathematics teacher tensions (20 minutes). Then, using our collective experiences with teachers, as researchers in Mathematics Education and/or as mathematics teacher educators, in small groups we will discuss the accuracy and/or limitations of the theoretical construct of teacher tensions in teaching and in the professional development of teachers (15 minutes). The subsequent 20 minutes will be dedicated to share out within the whole group the discussion about theoretical constructs vis-a-vis our own experiences. We will return to small groups for the next 15 minutes to construct the necessary elements of a working definition of (a) what teacher tensions are, and (b) what teacher tensions do. The remaining 20 minutes will be dedicated to whole group sharing and tentative conclusions.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 189-190). Umeå, Sweden: PME.

Slot 2 is dedicated to possible methodological approaches to gathering data on teacher tensions. We aim also at discussing the possible pedagogical approaches to invoke teacher tensions. We will start with a summary of the discussion from Slot 1, as well as a new definition of tensions constructed by the WG leaders from the work of Slot 1 (10 minutes). The next 15 minutes will be spent in small groups discussing methodological issues related to tensions (both data collection and data analysis), which will be followed by another 20 minutes of whole group share out. Again in small groups, pedagogical approaches to invoke teacher tensions will be addressed (15 minutes), and then shared out with the whole group (20 minutes). In the last 10 minutes of Slot 2, we will propose to establish a network of researchers willing to work on this over the next year.

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ARGUMENTATION AND FORMATIVE ASSESSMENT: A POSSIBLE SYNERGY?

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This working group is a new initiative, aimed at conjugating two research themes: formative assessment and argumentation in the mathematics classroom.

Formative assessment (FA) refers to educational assessment activities aimed at supporting learning and teaching, i.e. assessment activities *for* learning rather than *on* learning. According to Black and Wiliam (2009),

> practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited (p. 9).

Technology can offer great potentialities with this respect, as recent research and projects are showing (see the European project FaSMEd, which was aimed at investigating the role of technologically enhanced FA methods to support student learning, www.fasmed.eu; and Olsher, Yerushalmy & Chazan, 2016).

On the other hand, mathematical argumentation has been widely studied by scholars in the last decades. It can be conceived as "*the discourse or rhetorical means (not necessarily mathematical) used by an individual or a group to convince others that a statement is true or false*" (Stylianides et al 2016, p. 316). While some scholars focused on the possible links between argumentation and proving processes, others underlined the importance of classroom discourse as social activity and as a source of mathematical learning (see Stylianides et al., 2016, for an overview).

Within FaSMEd, we deepened the reflection concerning the value of argumentation practices as crucial formative assessment tools in the mathematics classroom (Cusi, Morselli and Sabena, 2017). The WG is aimed at promoting a complementary reflection on the dialectical relationship between formative assessment and argumentation and will involve participants in working together and sharing their research experiences around the leading question: "*How may formative assessment practices support mathematical argumentation?*"

The work with participants will start from our experience within the FaSMEd project and from the data coming from the project. Participants will be actively engaged in addressing the research questions, performing the analysis of the data according to their own theoretical lenses and experience. Indeed, a fundamental aim of the WG is to create conditions for possible networking of theories to study and to establish a

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 191-192). Umeå, Sweden: PME.

community of researchers interested in the theme, so as to pursue the collaboration after the conference and be able to organize a colloquium in PME within two years.

STRUCTURE OF THE SESSIONS

Participants will be involved in working on two connected issues: task design and methodology of work in classroom. Session 1 will engage participants in discussing specific features of argumentative tasks, in order to possibly outline categories of argumentative tasks. The leading research question will be: *How to design formative assessment tasks with argumentative components (also considering different school levels)?* Session 2 will focus on the methodology for the implementation of argumentative tasks in the classroom. Leading research question will be: *How to design a methodology of work in the classroom aimed at supporting the students' development of argumentative competencies through formative assessment practices?* The detailed outline of the two sessions is summarised in the following table:

Session 1	Session 2
• Introduction of the theme and brief sharing of interests between participants • Brief presentation of the FaSMEd project (theoretical framework and data) • Focus on task design: group-work activity on analysis of formative assessment tasks with argumentative components • Discussion among participants • Brief summary of key-results and issues emerging from the session.	• Brief recall of Session 1 and introduction of the main focus of the session • Introduction to the data that will be analysed • Focus on methodology: group-work activity on the analysis of transcripts of classroom episodes • Discussion among participants • Brief summary of key-results and issues emerging from the two sessions.

Table 1: The outline of the two sessions

References

Black, P., and Wiliam, D. (2009). Developing the theory of formative assessment. Educational Assessment, *Evaluation and Accountability*, 21(1), 5-31.

Cusi, A., Morselli, F., & Sabena, C. (2017). Promoting formative assessment in a connected classroom environment: design and implementation of digital resources. *ZDM Mathematics Education* Vol. 49(5), 755–767.

Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The second handbook of research on the Psychology of Mathematics Education* (pp. 315-351). Rotterdam, The Netherlands: Sense Publishers.

Olsher, S., Yerushalmy, M., & Chazan, D. (2016). How might the use of technology in formative assessment support changes in mathematics teaching? *For the learning of mathematics*, 36(3), 11-18.

INSTRUMENTS AND THE BODY

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The central goal of this working group is to study the possibility of bringing the body and embodiment into instrumental approaches to the use of digital technology in mathematics education. After two short introductions into instrumental views and the issue of embodiment, the participants will analyse a small data set using aspects of these perspectives. Opportunities and constraints of combining approaches for the sake of relating instruments and the body in mathematics education will be discussed. As a possible next step, the option of writing a joint research paper will be explored.

THEORETICAL BACKGROUND, TOPIC AND GOAL

Digital technology is omnipresent nowadays. This deeply affects education, and the teaching and learning of mathematics in particular (Monaghan, Trouche, & Borwein 2016). To address the complex relationship between the use of digital tools and the learning of mathematics, so-called instrumental approaches to tool use were developed since the 1990s (Artigue 2002; Drijvers, Godino, Font, & Trouche 2013). These approaches stress the intertwinement and co-emergence of the techniques to use specific digital tools and the schemes corresponding to the mathematical topic at stake. These instrumental approaches were helpful in going beyond the simple “just press the button” view on using digital tools, and addressing the subtlety involved.

A more recent development concerns embodied cognition (Lakoff & Núñez 2000). It stresses that cognition, even in the domain of mathematics, is rooted in bodily experiences, that take place in interaction with the world. Since then, the relationship between mathematics as material activity and the body has been exploited, giving rise to new views on embodiment (e.g. de Freitas & Sinclair, 2014; Ferrara & Ferrari, 2016). Beyond the tools and schemes central in instrumental approaches, these perspectives also include gestures, physical objects and arrangements (Ferrara & Sinclair 2016). Mathematical activity, in this view, involves the students’ bodies and other materiality.

The central goal of this working group is to study possible entanglements of these theoretical views for the case of using digital technology in mathematics education. What can each of the two contribute? Are they complementary or perpendicular? What happens if we short-circuit these two approaches? Can we give the instrumental approach the body back? How does the body come to matter in instrumental approaches? What is the relationship between mathematical concepts, body and the material activity with instruments? How do the different visions convey different assumptions on mathematical practice and mathematical concepts? These questions

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 193-194). Umeå, Sweden: PME.

guide the proposed working group, which is a new initiative. As a concrete outcome, the participants will investigate the possibility of publishing an article on this matter in a joint writing process after the conference.

PARTICIPANT ENGAGEMENT AND WORKING GROUP LAYOUT

As a preparation, participants will be asked to do some initial reading of some key publications in the field of instrumental approaches and new perspectives on embodiment.

In the first working group session, two 15-minute lectures will set the scene for the working group, one taking a primarily instrumental perspective, and the other taking an embodiment point of view. Next, the submitting researchers will take 10 minutes to present a small data set in the form of video data of students interacting with digital technology in a mathematical activity. The remaining 50 minutes will be devoted to a first exchange on this by participants divided into small groups.

During the first 30 minutes of the second working group session, participants will further analyze the data set in small groups. Each group will consider the data according to the questions above and will explore the opportunities and constraints that each of the perspectives offers. In the next 30 minutes, these experiences will be shared in the whole group. The final 30 minutes will focus on a discussion of possible next steps, in particular the exploration of a research paper in the line of the work done so far.

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REPLICATION IN MATHEMATICS EDUCATION

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The goal of this working group is to examine the role replication studies should have in mathematics education research. In recent years psychology has been gripped by serious methodological concerns: the so-called 'replication crisis'. A large number of psychology academics now believe that many published research findings cannot be replicated. This working group will discuss the replication crisis in psychology, discuss the extent to which similar issues apply in mathematics education, and consider potential solutions.

In 2011 Simmons, Nelson and Simonsohn noted that quantitative researchers have a great deal of flexibility in their analytical choices (e.g., exclusion criteria, dependent measures, including covariates), which may lead the traditional 5% false-positive rate to be inflated to as much as 60%. The most obvious consequence of an inflated false-positive rate is that some published scientific articles report effects that are simply not true. The Open Science Collaboration (2015), a group of 270 scientists, published a landmark paper that aimed to determine whether or not this was the case, by conducting 100 replications of studies reported in three psychology journals in 2008 97% of the original articles found significant results (i.e., p values less than .05), but only 36% of the replications did. Although there has been some debate about how to interpret the OSC's findings (e.g., Gilbert, King, Pettigrew, & Wilson, 2016), the headline figures do seem to suggest that the serious concerns are merited.

This issue is particularly live in the context of education research. Makel and Plucker (2014) found that only 0.13% of articles published in the top 100 education journals have reported replication studies. We simply do not know the extent to which published educational research findings replicate.

The goal of this working group, which will meet for the first time at PME42, is to focus on the role of replication research within mathematics education. Drawing on recent discussions in the *Journal for Research in Mathematics Education* (e.g., Schoenfeld, 2018; Star, 2018), we will discuss the following topics:

- What is replication research in the context of both quantitative and qualitative mathematics education research?
- Should replication studies be conducted, presented at scientific conferences and published in high-profile academic journals?
- Should the mathematics education community encourage actions to improve replicability (e.g., preregistered analysis plans, open data, etc.)?

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 195-196). Umeå, Sweden: PME.

Based on the discussions, a Research Forum may be set up for the forthcoming PME conference, and a special issue focussing on good replication practices in mathematics education may be initiated.

ACTIVITIES

First Session (90 min): What? And why?

- Introduce the goal of the WS and presentation of examples of (quantitative) replication studies in mathematics education (20 min).
- Small group discussion: do (quantitative) replication studies in mathematics education have value? (25 min). Followed by plenary summary (20 min).
- Plenary discussion on replication in qualitative research (25 min).

Second Session (90 min): How?

- Summary of results of the first session (10 min).
- Small group discussion: quality criteria for replication studies and guidelines for their inclusion in conferences and journals (30 min). Followed by plenary summary (20 min).
- Plenary discussion on future ideas (research forum, special issue) (30 min).

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INTERNATIONAL PERSPECTIVES: MEASURING MATHEMATICS TEACHERS' KNOWLEDGE IN THE DIGITAL ERA

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In this working group, the international group of researchers will explore current issues related to the design and development of measures of mathematics teachers' knowledge in the context of mathematics teachers' change. The goal is to examine current strategies used by researchers to design measures of different types of knowledge needed for teaching mathematics. Researchers use a wide range of conceptual and theoretical frameworks to conceptualize and develop measures of different types of teachers' knowledge. The discussion of theoretical and methodological challenges associated with the design of measures will also be included in this working group discussion led by a diverse group of researchers.

OUTLINE OF THEORETICAL BACKGROUND

Design of assessment tools for measuring different types of mathematics teachers' knowledge is based on two main factors: purpose and content (Orrill, Cohen, 2016). There are different conceptual frameworks for describing mathematical knowledge needed for teaching (Hill, Ball & Schilling, 2008; Kaarstein, 2014; Kaiser et. al., 2014; Manizade & Martinovic, 2016; Manizade & Mason, 2011; Shulman, 1987; Silverman & Thompson, 2008; Tirosh, 2000). Any assessment has to be based on careful consideration of the domain. Researchers have worked tirelessly to develop measures of teachers' knowledge for teaching mathematics and its related constructs (Herbst & Kosko, 2014; Hill et al., 2008; Kaiser et al., 2014; Manizade & Mason, 2011; Silverman & Thompson, 2008; Thompson, 2016; Tirosh, 2000). This WG is based on the aforementioned theoretical framework.

WORKING GROUP AS A NEW INITIATIVE

The purpose of this working group is to provide a platform for sharing international perspectives on measuring mathematics teachers' knowledge for teaching different strands of mathematics. Researchers across continents have been developing various types of measures of mathematics teachers' knowledge using a range of theoretical frameworks and methodological approaches. Our goal is to start a dialogue amongst these researchers, provide a critical review of currently existing measures, and discuss strengths and limitations associated with each approach. By doing so, our intent is to identify future paths for research on the design and development of measures of mathematics teachers' knowledge in the digital era.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 197-198). Umeå, Sweden: PME.

PROPOSED LAYOUT AND OPPORTUNITIES TO CONTRIBUTE

Below is the figure describing the layout of the session and presenting authors. However, all interested PME participants are invited to attend and contribute to this working group as well as engage in the discussions highlighted here.

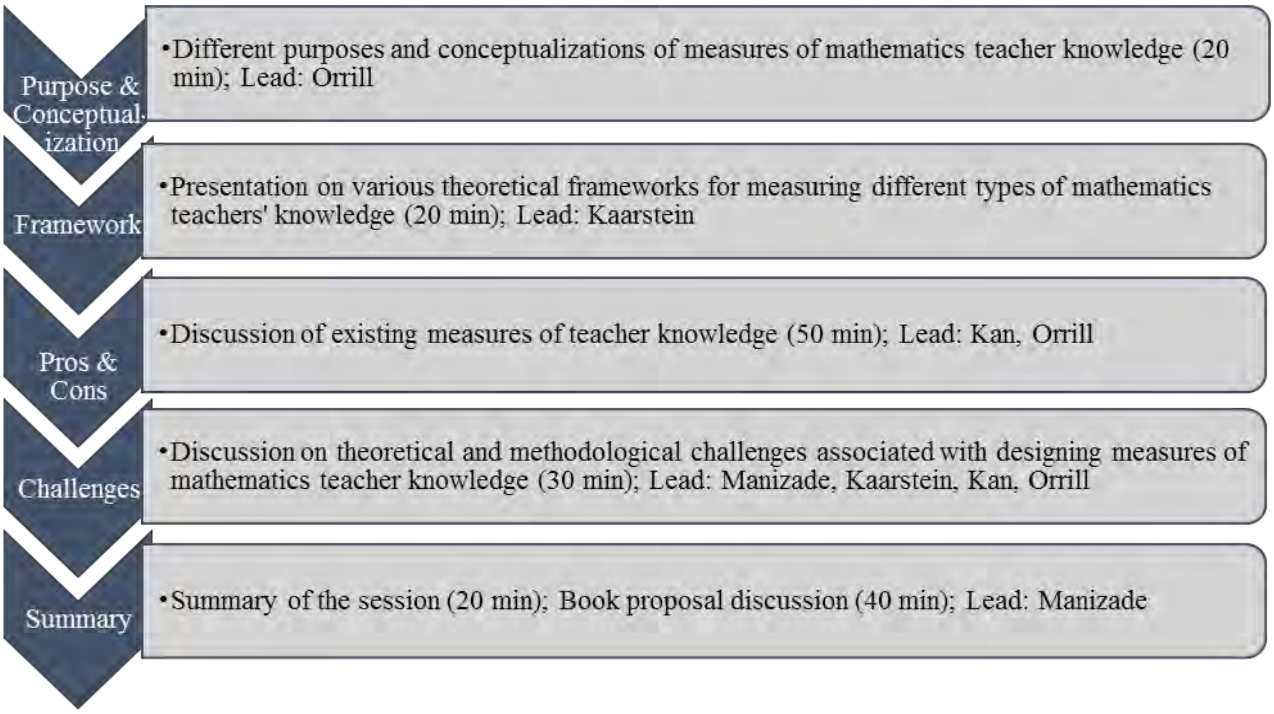

Figure 1: Proposed Layout

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EXPLORING THE ROLE OF FACILITATORS IN VIDEO-BASED PROFESSIONAL DEVELOPMENT FOR MATHEMATICS TEACHERS

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Video is a central tool in professional development programs for mathematics teachers worldwide, and its various uses have been extensively studied. One of the core aspects of video-based programs, which to date has been less investigated through empirical research, is the role of facilitators in enhancing effective teacher learning. The working group will focus on this issue and explore it from various perspectives, building on the collective input of the group participants in forming relevant and re-searchable questions as well as approaches to study them.

BACKGROUND

Video-based professional development (PD) programs for mathematics teachers have become prevalent as means to support teachers' professional growth (Karsenty & Sherin, 2017). In a previous Working Session held at PME 41 (Karsenty, Coles & Hollingsworth, 2017) participants have discussed and categorized several existing frameworks for using classroom video in PD scenarios. One of the questions raised in the second session of this WS concerned the role of the PD facilitator and what is entailed in this role. Empirical research on PD facilitators in general, and on video- based PD facilitators specifically, has begun to gradually accumulate only in recent years (e.g., Borko, Koellner & Jacobs, 2014; Coles 2014, 2016; Karsenty, 2016; Lesseig et al., 2017; van Es et al., 2014). The importance of this emergent domain of research lies in the recognition that within any attempt to reach a sustainable and scalable PD model, it is critical that facilitators would be able to adapt the model to various contexts while maintaining integrity to its original goals and agenda (Borko et al., 2014). Thus, the role of facilitators is closely linked to issues such as the knowledge and awarenesses needed for conducting PD within a specific model, the types of pedagogies and analyses that are seen as productive, and the desired community norms.

GOALS

1. Discuss emergent knowledge on the role of facilitators in video-based PD contexts, as formed by existing literature and by current experiences of the WG participants.
2. Raise key questions for relevant future research.
3. Elaborate on possible methodologies for investigating these key questions.

ACTIVITIES AND TIMETABLE

Session (1):

- Introduce session aims and provide overview of existing literature. (15 min)
- Watch a videotaped episode of authentic facilitation (filmed in a PD program for secondary school mathematics teachers) and discuss the role of the facilitator as

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 199-200). Umeå, Sweden: PME.

interpreted by the WG participants (whole group discussion). (45 min)

- Given the common experience of the video, participants will share perspectives, in small groups, regarding the following questions: What characterizes effective facilitation of teacher discussions around video? What kind of knowledge and skills may be needed for effective facilitation, and how may these be acquired? (30 min)

Session (2):

- Review work from previous session: Each group will present a summary of their work in Session 1 (this also allows access into the group for newcomers) (15 min)
- Based on this review, discussion in small groups will continue, focusing on the following task: List several research questions that interest you, in the area of facilitation of video-based PD. Which of these questions are researchable in the contexts known to you? What methodologies can you suggest for studying these questions? Can you think of theoretical/conceptual frameworks that would be helpful when analysing the collected data? (40 min)
- Groups share and discuss their input in the plenary. (25 min)
- Discuss next steps for future collaborations. (10 min)

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LEARNING MATHEMATICS IN/THROUGH/BY ARTS PRACTICES

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[1]Manchester Metropolitan University, [2]Simon Fraser University

This Working Group will use material experimentation and discussion to outline a research program encompassing the historical, cultural and educational connections between mathematics and the arts. Through experimentation with sculptural crochet techniques, participants will pursue the bodily experiences and philosophical questions that arise from a mathematical engagement with art making. We will reflect on how these questions might constitute a ground for a research program to be pursued by mathematics educators and researchers. The sessions will be oriented towards envisioning lines of collaborative work among the participants and subsequent writing of a joint paper articulating major research questions about learning mathematics and the arts as entangled and embodied practices.

WORKING GROUP GOALS

1. To experiment with artistic practices taken from the field of fiber art
2. To explore how art practices stand in conversation with mathematical ideas
3. To discuss the experience of making, critically engage with the philosophical questions that arise from this experience, and begin to outline a research program on learning mathematics and the arts

OUTLINE OF SESSION ONE

I. Learn to crochet simple three-dimensional objects and reflect on these experiences (30 min)

II. Presentation with a focus on the mathematics arising from art practices (20 min)

III. Discussion (40 min)

Experimental play with crochet techniques can give rise to mathematical ideas and, vice versa, artistic techniques also arise from the application of mathematical concepts. Participants in this seminar will first learn the basic crochet knot and be given time to explore its sculptural possibilities. As participants dig into this work, we will share examples of the mathematical explorations that have arisen from the material practices of artists. This includes both the formal mathematical modelling of already existing cultural productions, of which perspectival geometry and crochet are examples (see Gamwell, 2015, Osinga, & Krauskopf, 2004) and the embrace of types of mathematics that arise from the material traditions of non-western or minority cultures, which is often advocated by the field of ethnomathematics (see Gerdes, 2011; Zaslavsky 1991). The presentation will open up to a discussion of the political and philosophical distinctions between craft and art, tradition and modernity, concrete and abstract: In what

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 201-202). Umeå, Sweden: PME.

sense does art involve abstraction? How is mathematics related to concrete and material practices? We will use the remaining time to discuss these questions and strive to elaborate on them in connection with our crochet work.

OUTLINE OF SESSION TWO

I. Learn to create sculptural objects with crochet techniques and wire, and reflect on these experiences (30 min)

II. Brainstorm about major research questions on learning mathematics and the arts (20 min)

III. Discussion towards writing a joint paper outlining a research program (40 min)

We will continue the material experimentation from the last session by adding wire to our crochet work. We will examine art objects generated through an open-ended exploration of mathematical concepts by looking at the work of mathematician-artists who have contributed to art exhibitions and public art (see Yackel, 2016). This mode of working will also be illustrated by the projects of students whose teachers engage them in art production as means of expression for the insights they experienced in their mathematics classes (Ernest & Nemirovsky, 2016). We will speak about ongoing work that seeks to be immersed in a dynamic and mutually productive conversation between arts practices and mathematical inquiry (Nemirovsky, 2018). The presentation will be followed by a broader discussion about the lines of research that participants want to pursue to address central issues that entangle the learning of mathematics and the arts. Plans will be discussed for a subsequent joint paper to be submitted to the Journal of Mathematics and the Arts.

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MATHEMATICS EDUCATION RESEARCH FROM AND IN LATIN AMERICA

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[1]Federal University of Mato Grosso do Sul, [2]Roskilde University, [3]Autonomous University of Madrid

The goal of this group is to gather Latin American researchers and find a common understanding about the school mathematics context in this region from a socio-political perspective on mathematics education. The strategy involves exploring PME 42's topic "delight in mathematics education" evoking resembling particularities among Latin American countries as examples of psychological manifestations of the socio-political in school mathematics.

PARTICIPANTS

Besides main contributors, other researchers have manifested their interest: Paola Valero from Colombia; Marcio Silva, Marilena Bittar, Rute Borba and Gilda Guimarães from Brazil; Melissa Andrade-Molina and Alex Montecino from Chile.

THEORETICAL BACKGROUND

School mathematics has been taking an important role in society. The statement about the importance of "mathematics for all" is sustained by the idea that "one such attribute of a modern human being is her ability to understand and master the world. Mathematics seems to lend us such divine powers" (Lundin, 2012, p. 07). However, it was not always like this, as argued by Yolcu (2017, p.15). In this sense, authors like Popkewitz (2004), Valero (2017) have problematized the effects of subjectification in the production of the desirable citizen: "school has connected the scope and aspirations of public powers with the personal and subjective capacities of individuals" (Popkewitz, 2004, p. 07). Also, the development of mathematical skills in children is understood as essential to the development of nations in the logic of modernity: "Math skills are proven to be fundamental to a person not only as a skilled workforce, but also as a citizen", to achieve "social progress, economic growth, and citizenship (Valero, 2017, p. 123). Our argument is that these cadres link socio-political perspectives that produce "an alchemy" (Popkewitz, 2004) with specific psychological inscriptions in Latin American students.

FIRST SLOT

Opening presentation: Delight in Mathematics Education (Bruno, 10 min.) We propose a reflection on this theme as a psychological inscription (Popkewitz, 2004) and on its socio-political implications.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 203-204). Umeå, Sweden: PME.

Discussion opening: Subjectivation in Brazilian rural textbooks (Neto, 15 min.) An illustration of effects on the subject via mathematics education as a political technology.

Discussion (55 min.) All invited researchers will discuss in groups and provide a summary of how they are understanding "psychological inscriptions" in mathematics students in each of the countries represented.

Wrap-up (10 min.) Focusing first in particularities of each country and common elements, we leave the open question about what is it different about each region.

SECOND SLOT

Discussion opening: The Chilean laboratory: from the new math to PISA (Elicer, 15 min.). An example of progressivist ideas "incepted" in Chilean school mathematics from international agendas.

Group work (50 min.) Researchers are invited to join a researcher from different nationalities. Each pair will draw up a common report in order to describe how the alchemy described by Popkewitz (2014) is drawn in their respective experiences and generates "psychological inscriptions" in mathematics students.

Closing discussion (25 min.) We go back to the topic of "delight" in mathematics education problematized as a psychological inscription. Contributors take notes for sketching main ideas for the report. The aim of continuing common work to think the Latin American context is proposed.

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ASSESSMENT FOR LEARNING IN DIVERSE CLASSROOMS

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The guiding principle of assessment for learning (AfL) is that students should understand how and why they are assessed, they should learn from the assessment what they have achieved or learned so far and they should get feedback that helps them move their learning forward (Wiliam, 2007). Students might be assessed by their teachers, by their peers or they might self-assess. In AfL classrooms, the ideal is for teachers and students to share responsibility for learning. This may necessitate changes in their thinking or meta-cognitive awareness regarding assessment and classroom discourse. Shared responsibility can only be possible if the teaching is student-centred and if students are empowered to take responsibility for their own learning.

Across the world, classrooms are becoming more diverse, encompassing students at different levels of ability and a wider range of socio-economic backgrounds. Due to increased migration, diversity has often also come to encompass language and culture. This adds to the challenges faced by teachers when attempting to adapt teaching and assessment activities to their students. We propose that AfL is challenging, because teachers need to be sensitive to students' varying needs. Research literature focusing on AfL in diverse classrooms is scarce. Research on AfL practices often describes them in general terms without regard to diversity in background, culture, language or ability. Adaptation of feedback to diverse students is therefore seldom dealt with. In order for feedback to enhance learning, students need to understand and make sense of not only feedback but also the assessment criteria. This is why teachers need to consider learners' different ways of participating (Anthony & Walshaw, 2009; Bruno, Santos, & Costa, 2016; Dann, 2015).

The aim of this working group is to contribute to the understanding of challenges connected to adapting AfL practices to diverse classrooms by discussing the theoretical underpinnings of AfL, relating them to research on diverse classrooms and differentiation. The two sessions will be used for group discussion. Nortvedt will act as coordinator.

SESSION 1: THEORETICAL UNDERPINNINGS OF AFL

The first session will start with a short introduction to the workshop in which we will problematize some of the underpinning theoretical constructs of Afl and relate these to diverse classrooms (15 min). Following this, participants will work in small groups to discuss the meaning of assessment, in general and in AfL, based on statements taken from the introduction (20 min). The statements will be related to AfL, diversity and the process of differentiation. All groups will be asked to share their thoughts with other

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 205-206). Umeå, Sweden: PME.

participants in the working group. We will use these thoughts to create a mind map and link these to contributions from previous research and theoretical reflections of AfL (15 min).

The last 30 minutes of the first session will be used to discuss in groups the meaning of good feedback within different approaches to teaching mathematics. There will be 15 minutes of small group discussion followed by 15 minutes of whole group discussion.

SESSION 2: ADAPTING AFL TO THE NEEDS OF DIVERSE STUDENTS

In the second session, we will discuss what it means to adapt feedback to diverse students. In classrooms where students of different ability levels receive mathematics teaching together, not only the teaching but also the feedback provided by the teacher will need to be adapted to individual students (Anthony & Walshaw, 2009). How can this be understood and researched? How can mathematical activities be adapted to students at different achievement levels in order to create opportunities for self-assessment? After a short (5 min) introduction, 40 minutes will be used for a group discussion of how mathematical classroom situations might be designed to facilitate differentiation. There will be 10 minutes of group brainstorming to create scenarios to be explored in small groups, 20 minutes of small group discussion of these scenarios and 10 minutes of full group discussion.

Diverse classrooms might also be multicultural. Within different cultures, students might hold different beliefs about what it means to learn mathematics and consequently, they might relate differently to teacher feedback (Moschkovich & Nelson-Barber, 2009). The last 45 minutes of the second session will be devoted to a group discussion of how AfL might be understood and defined to take into account some of the challenges students and teachers might face in multicultural and diverse classrooms.

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MATHEMATICAL LEARNING DISABILITIES: A CHALLENGE FOR MATHEMATICS EDUCATION

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CONTEXT

These last decades are clearly marked, at the international level, by an increase of research and a better comprehension of learning disabilities. Some of them remain complex and less studied (Lewis & Fisher, 2016). That is the case of *mathematical learning disabilities* (MLD) which are the source of raising educational and social inequalities. However, definitions of MLD remain elusive. MLD are often reduced to difficulties in processing numerical quantities and arithmetical calculations. In particular, they do not take systematically into consideration specific difficulties in mathematical reasoning. Recent publications point out that MLD are heterogeneous (Fias, Menon & Szũcs, 2013) and affect several aspects of mathematical skills (Kaufmann & al., 2013). Research regarding MLD is carried out in different fields, with various theoretical backgrounds, hypothesis and aims (Butterworth et al., 2011; Lewis & Fischer, 2016): cognitive sciences, neuroscience, psychology, mathematics education. Nowhere it appears a clear consensus about the definition of the MLD and about their diagnosis. Moreover, the links between these different fields of research are not obvious and should be improved. Our team (called RITEAM for "Recherche Internationale sur les Troubles dans l'Enseignement et l'Apprentissage en Mathématiques" i.e. "International Research on Learning Disabilities in Mathematics Teaching (see riteam.ch) consists of five researchers in mathematics education coming from several countries (France, Switzerland, Italy, Canada) where politics are concerned by the difficulties in the learning of mathematics and the processes of "inclusion" (e.g. "loi pour la Refondation de l'École", 2013 in France; "Loi 170/2010 and Dir. 27/12/2012" of MIUR (2010 & 2012) in Italy; "De l'intégration à l'inclusion scolaire des élèves en difficulté d'adaptation et d'apprentissage", CTREQ (2009) in Québec, Canada; "Accord intercantonal sur la collaboration dans le domaine de la pédagogie spécialisée", 2007 in Switzerland). We claim that specific studies should be structured and developed in mathematics education regarding MLD in order to improve the identification and the remediation of MLD in an educational context. In particular, that implies a better knowledge of the existing research dealing with MLD. That is the reason why our team is working on these two main aims:

- To circumscribe research about MLD in mathematics education and to federate new collaborations in this field;

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 207-208). Umeå, Sweden: PME.

- To structure a collaboration at the interplay between mathematics education and cognitive sciences: we hope that such collaborations will evolve (following for instance De Smedt & Verschaffel (2010), Gardes & Prado (2016)).

PLANNED ACTIVITIES

The activities of this WG will be organized around the above-mentioned aims. We believe that research done in collaboration between several mathematics educators coming from countries over the world is a powerful tool for structuring and federating new trends regarding MLD in mathematics education. During the first session, we will present RITEAM and list major publications in the field (30 min). Then, we will break up into smaller groups in order to identify several current and future problematics about MLD (45 min). We will conclude and mutualize the results (15 min). The second session will start by the presentation of a free reference management software (15 min). Main ideas from the first session will help to build a structure in this software (definition of keywords, inventory of pertinent journals, identification of theoretical backgrounds, etc.) (1h). Such a classification and database can evolve during the two sessions, with the participants' experience in the field of MLD and/or mathematics education and through discussions and analysis in small groups and in plenum. The WG will end by concluding remarks and a definition of a future collaboration between the participants (15 min).

OUTCOME

The working session will bring a first collaborative inventory and classification of publications regarding MLD in mathematics education. The aim of this mutual tool is clearly a way to federate new collaborations in the PME community about education questionings and research regarding MLD.

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EYE-TRACKING IN MATHEMATICS EDUCATION RESEARCH: A FOLLOW-UP ON OPPORTUNITIES AND CHALLENGES

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Involvement of eye-tracking (ET) in educational research is growing in recent years (van Gog & Scheiter, 2010; Was, Sansosti, & Morris, 2017) and ET equipment and technology have become more affordable. Already at PME 38, there was a Working Session on *The use of eye tracking technology in mathematics education research* by Barmby, Andra, Gómez, Obersteiner, and Shvarts, which served an introductory purpose and investigated possible research questions that would benefit from ET research methodology. However, over the past four years the technology and its use have intensively developed; on-line ET by ordinary web cameras is promising to become a part of everyday e-learning user experience in the next few years. Currently, there is new ET technology available on the market and being developed (e.g., Toivanen et al., 2017), which offers new opportunities but also presents researchers with certain challenges regarding setups of studies and methods of analysis. Besides, one of the most promising developments in ET research is dual and multiple ET: the analysis of two or more persons' eye movements that allows for studying collaboration and the teaching/learning process in vivo (Lilienthal & Schindler, 2017; Shvarts, 2018). Still, dual and multiple ET are challenging—not least on a technical level.

The ET sub-community is just at the beginning of its development, however there are quite a lot of independent research groups who conduct such studies. This WG aims to attract attention of researchers (1) who are interested to use ET in future research and are interested in information on current ET opportunities and challenges, (2) who would like to use ET technology but experience a lack of access to the equipment, and (3) who are currently involved in ET studies in mathematics education for collaboration and exchange purposes. Accordingly, the *goals* are as follows.

1. To grow awareness of opportunities and limitations provided by ET technology, including innovative methods of data collection and analysis.
2. To make ET technology more accessible for researchers—e.g., by sharing experience of low-cost eye-tracker's usage, by providing hands-on possibilities to test different ET systems, and by encouraging possible collaborations.
3. To strengthen the exchange and collaboration between people who are or have been actively conducting ET research in mathematics education. We seek for planning future group activities, e.g., at PME or in scientific journals.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 209-210). Umeå, Sweden: PME.

Table 1 gives on overview of the planned schedule and the activities connected to the three goals of the WG.

	Duration	Activity
Session 1	10 min	*Impulse presentation:* Opportunities and challenges of ET technology
	20 min	*Discussion in groups:* Possible areas of ET research, theoretical backgrounds and research questions
	10 min	*Impulse presentation:* Dual/multiple ET : collection and analysis.
	20 min	*Discussion in groups:* Exchanging experiences with dual/multiple ET, preferences to conduct dual/multiple ET studies (equipment, analysis), encountering challenges, etc.
	30 min	*Open Exchange Forum (OEF):* (Aim 1) Collecting further questions and discussing possible future ET projects; (Aim 2) Discussing of project ideas and opportunities of ET, finding ideas for brief records in Session 2; (Aim 3) Gathering of experiences in ET research and possible collaboration partners
Session 2	10 min	*Impulse presentation:* Summary of Session 1
	30 min	*Work in groups (drawing on OEF):* (Aims 1 & 3) Discussing and presenting samples of ET data in math education; (Aim 2) Recording sample data in groups with different ET systems to test ET recording quality and procedure
	30 min	*Work in groups:* (Aim 1) Sharing/discussing benefits and challenges of ET and project ideas; (Aim 2) Brainstorming/planning on possible future research projects, encountering the challenges, and initiating collaborations; (Aim 3) Gathering ongoing ET research in math education & planning for future group activities (PME, Journals, etc.)
	20 min	*Plenary discussion:* Sharing results of the discussions and summarizing the outcomes of the WG

Table 1: Schedule and activities of the WG.

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CONSIDERING THE DESIRED TEACHER OF MATHEMATICS TEACHER EDUCATION

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The purpose of this new working group is to work collaboratively towards characterisation of images of the desired teacher reflected in teacher education programmes around the world.

We pursue this aim mainly through activities. Hence, we encourage interested participants to bring assessment criteria, assessment tasks, observation protocols, course & program descriptions, and teaching materials from their own institutions.

BACKGROUND

Teacher education generally has both a contextual component of observing and teaching in schools, and decontextualized elements in university courses (cf. Tatto et al., 2012). University courses may be content courses, general pedagogical courses, or mathematics education courses. The assessment criteria of each component or course transmit, both explicitly and implicitly, notions of the desired teacher.

Our key concept is the *desired teacher*. This recognises the teacher as both a subject and as subjected to discourses of power which constitute categories such as 'teacher' and 'good teacher' (cf. Montecino & Valero, 2015). Notions or images of the desired teacher produce 'regimes of truth' which normalise reflection - or not, normalise transformation of content as part of teaching - or not, etc. The notion of what the good teacher is, has varied over time and place. Images of the *charismatic teacher* with the *disposition* to teach, have been around for a long time, and still prevail in popular culture (Connell, 2009). Connell also identified a *technical-professional* model, which prevails in descriptions of teachers' "technical know-how". But what are current, competing, contrasting or co-existing images of the desired teacher?

This gives rise to this question guiding the work in the group:

- What are the images of the desired teacher reflected in references to desired knowledge and valued application practices, in institutional materials etc.?

STRATEGY FOR THE TWO SEMINARS

Activity 1: Introduction (20 min)

We start by giving some examples of the desired teacher reflected in the practicum observation protocols we have analysed. This is followed by a short discussion with participants around the question guiding the working group, possibly reformulating it.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 211-212). Umeå, Sweden: PME.

Activity 2: Ideas for analytical frameworks (30 min)

We invite participants into a dialogue about which variables should be considered in the analysis of mathematics teacher education materials. As part of this, we will give a short presentation of the framework we have utilised, starting from a distinction between whether the *valued application* is of (a) *reasoned judgement* or (b) *preferred technique* (Rusznyak & Bertram, 2015).

Strong arguments have been presented for the importance of professional judgement in teaching. Within this perspective, teaching is viewed as a principled practice requiring specialised knowledge. The distinction between the two perspectives is one of the relations to specialised knowledge, where the 'preferred technique' does not explicitly refer to knowledge which may have informed the technique.

Activity 3: Collaborative explorative analysis of materials (40 min + 35-45 min)

In smaller groups, the materials brought by the participants and/or us will be analysed using more or more of the analytical frameworks from Activity 2.

Interlude: Results of a creative engagement with our data (15 min)

To inspire further analysis, we may decide to give a short presentation, briefly interrupting activity 3, of our suggested images of the desired teacher: the knowledgeable teacher, the constantly improving teacher, the successful teacher, the knowledge-transforming teacher, and the inspired teacher.

Activity 4: Summarising and planning further work (30-40 min)

Hopefully, enough has come of the activities that some ideas can be summarised and possible collaborations after PME formulated.

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WORKING GROUP: THE DESIGN OF INTENDED MATHEMATICS CURRICULA

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This Working Group aims to engage PME participants in discussion and debate about principles for the design of 'intended' curricula, i.e. documents such as the Common Core State Standards in the USA and the National Mathematics Curriculum in England. Such documents set out expectations about what mathematical ideas should be taught and when and include learning goals to be met and assessed. By considering relevant methodological and theoretical advances in the field, together with two specific design efforts focused on intended curricula as contexts for discussion, we will aim to identify some design principles pertaining to the development and organization of intended curricula, the relationship between intended curricula and learning/teaching, and the cultural dimension of intended curricula.

THEORETICAL BACKGROUND

The term *curriculum* has been used in multiple ways in the literature and in different jurisdictions. This newly established Working Group focuses on a particular kind of curriculum that Schmidt et al. (1996) called the *intended* curriculum and broadly defined to include an educational system's visions, aims, and goals for students' learning of a particular subject. The notion of intended curriculum is akin to Remillard and Heck's (2014) notion of *official* curriculum, which, however, was framed more broadly to include also any curricular resources (e.g., textbook series) designated by an educational system as embodying its curricular vision. Given that many educational systems do not designate such instructional resources, we opt for the narrower notion of intended curriculum as the focus of this Working Group.

Schmidt et al. (1996) distinguished the intended curriculum from the *implemented*, which denotes the practices, activities, and broader institutional arrangements in schools or classrooms that are put in place within the educational system to enact its curricular vision. The intended and implemented curricula are mediated by a number of factors, such as the teachers' pedagogical orientations and interpretations of the intended curriculum or the instructional resources developed to translate the intended curriculum into classroom activities (Remillard & Bryans, 2004). Students' learning outcomes, the *attained* curriculum (Schmidt et al., 1996), are influenced by what the educational system intended students to learn but also by how those intentions were interpreted or translated, and ultimately implemented and experienced.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 213-214). Umeå, Sweden: PME.

GOAL AND ORGANIZATION OF THE WORKING GROUP

There are many reasons why intended curricula cannot be based solely on research (Hiebert, 2003). Yet the design of certain aspects of intended curricula can, and possibly should, be informed by research. The *goal* of the Working Group is to consider possible approaches to the design of intended mathematics curricula and to identify some design principles by drawing on relevant methodological and theoretical advances in the field. The aspects of intended curricula to be considered will inevitably be selective, but we hope that the discussion will be carried over in future conferences. In this conference we will focus on the following *three aspects* of intended curricula (a few illustrative issues/questions are included for each aspect):

1. *Development and organization of intended curricula:* How do intended curricula best present and communicate the relative importance of mathematical ideas to be learned by students, and relationships between different ideas?
2. *Relationship between intended curricula and learning/teaching:* How do intended curricula best present and communicate learning goals – As statements? As learning progressions? As learning trajectories? As growth points? What role, if any, should intended curricula play in informing preferred modes of pedagogy?
3. *Cultural dimension of intended curricula:* In what ways does culture interact with mathematics and affect curricular goals? Are there any parts of a mathematics curriculum that may be considered to be core or universal? What principles might guide cultural adaptations?

We will begin with two presentations (15 mins each) on two design efforts focused on intended curricula: first, by William G. McCallum, on the US Common Core State Standards (http://www.corestandards.org/); second, by Lynne McClure, on Cambridge Mathematics (http://www.cambridgemaths.org/). The presentations will offer a specific context for our discussion of each of the above three aspects (40 mins per aspect). Working Group participants will be expected to participate in the discussion, sharing practices from their educational systems. The last 30 mins will be a general discussion of emerging design principles and future work.

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MATHS AND SPECIAL EDUCATION WORKING GROUP

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This working group, active during PME 38-40 and PME-NA 34-39, has been developing a research agenda to explore pedagogical approaches for fostering conceptual knowledge of mathematics in students with special learning needs in mathematics. Our work has been rooted in a twofold premise: (a) students with special needs are capable of and need to develop conceptual understanding and mathematical reasoning skills, and (b) special education instruction, assessment, and research needs to transition towards this focus. In this working group, the focus will be on publication of research findings through cross-disciplinary collaboration.

BACKGROUND

About five to ten percent of school-age children have been identified as having mathematics disabilities (Fuchs, Fuchs, & Hollenbeck, 2007) and students whose mathematics performance was ranked at or below the 35th percentile are often considered at risk for having learning difficulties in mathematics (Bryant et al., 2011). The purpose of our working group is to explore issues of research around the intersection of mathematics education and special education. Substantial work exists that focuses on the development of conceptual understanding and mathematical reasoning of students in general education. However, much less is known about the mathematical reasoning of students with disabilities or how to support the development of their conceptual understanding. The absence of research addressing this subset of students may be due in part to theoretical orientation of the field of special education, which emphasizes explicit teaching of targeted skill-sets (Gersten et al., 2009). In addition, it seems that many special educators consider "research-validated" interventions to largely include studies that follow large-scale, Randomized Control Trial approaches (Woodward & Tzur, 2017). Often, the role of the pedagogical content knowledge of the interventionists or how the intervention helps students construct the concept or develop mathematical reasoning are lost in large scale nature of these types of studies.

Since 2008, members of this working group have been collaborating on research projects that integrate research-based practices from mathematics education and special education. This proposed working group will continue to promote this collaborative, cross-disciplinary work, in particular by strengthening understanding between international researchers and between practitioners of mathematics education and special education.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 215-216). Umeå, Sweden: PME.

PLAN FOR WORKING SESSION

During the previous three meetings, we discussed our intention to further explore possibilities for collaborative work in this topic, including proposing and planning for an edited book on mathematics and special education. Table 1 presents a work plan to continue this collaborative publication effort.

Session 1	Session 2
Introductions, including attendees' work pertinent to mathematics and special education	Assign chapters
Briefly share history of working group	Write abstracts for the chapters
Review discussion at PME-NA 39 about book proposal	Revise book description
Revise unique selling points of book	Identify tasks for the members of the working group to achieve during the year
Decide on format for book	
Outline desired chapters	

Table 1: Goals and activities for the working group

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SEMINAR

REVIEWING FOR PME
A PRIMER FOR (NEW) REVIEWERS

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BACKGROUND

Seminars are intended to provide specific courses for professional development of PME members.

Any PME member with two accepted Research Reports in the last five years, or three accepted Research Reports in the past ten years, is eligible as a reviewer for PME conferences.

GOAL OF THE SEMINAR

This seminar is intended to provide information about the PME review process and give the opportunity to gain first experiences in providing a high-quality review. The seminar aims especially at the needs of new reviewers, although experienced reviewers are highly welcome in order to facilitate knowledge transmission within the PME community.

The seminar includes an introduction to the intention and purpose of reviewing from a more general perspective (McKnight et al., 2000; APA, 2009), but also details aspects of the PME review practices. Participants will have opportunities to work with authentic examples from the PME review processes of recent years. Acknowledging the diversity within the PME community in the review process will be an important aspect of the seminar.

GOALS FOR THE PARTICIPANTS

Having participated in the seminar, the participants will:

1. know about reviewing as an aspect of scientific quality management,
2. know about the most important differences in reviewing procedures for journals and conferences as well as different types of contributions, especially in the PME context,
3. be able to differentiate the specific review categories of PME,
4. be able to identify aspects of quality for a review, and
5. be sensible to aspects of fair, constructive, and inclusive reviews.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 219-220). Umeå, Sweden: PME.

EXPECTED BENEFIT FOR PME AS A COMMUNITY

PME – as a scientific community – will benefit from the seminar as:

- It is expected to improve the knowledge of (new) reviewers about the review process.
- It is expected to smoothen (new) reviewers' difficulties in composing high-quality reviews.

METHODS

The seminar will be developed throughout two sessions of 90 minutes each. It will start with a brief presentation focusing on learning goals 1 and 2. A first group work phase will focus on the specifics of PME reviews and will thus contribute to learning goals 3 and 4. A second group work phase will focus on the aspects of fair, constructive, and inclusive reviews (learning goal 5). Experienced reviewers, who are willing to share their knowledge, are invited to serve as group mentors during the working phase.

If you are willing to share a former contribution of yourself in addition to the reviews you will receive as authentic examples for the group work phase, please contact David M. Gómez at david.gomez@uoh.cl as soon as possible.

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COLLOQUIUM 1

PARALLEL ANALYSES OF COLLABORATIVE MATHEMATICS PROBLEM SOLVING IN A LABORATORY CLASSROOM SETTING

Organizer: Man Ching Esther Chan[1], Discussant: Markku Hannula[2]

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Learning in social settings such as the mathematics classroom involves complex processes combining cognitive, affective and social elements. *The Social Unit of Learning* project is an international collaborative project that aims to understand learning as a social phenomenon and, particularly, how it occurs in the mathematics classroom. Collaborative problem solving is used as a way to create situations whereby students have to negotiate meaning overtly in a social setting. Using advanced video technology and purposefully designed classroom activities, the data collected for the project offer the opportunity for structured, rigorous, fine-grained investigation of the social aspects of collaborative problem solving and learning.

Researchers from Australia, Spain, and Finland have undertaken parallel, complementary analyses of a shared dataset to investigate the complexity of collaborative mathematics problem solving and learning. Data were collected using a laboratory classroom with multiple cameras and audio inputs. The research design created situations requiring individual, dyadic, and small group (4-6 students) problem solving in mathematics and documented the social interactions and associated learning. A multitheoretic research design utilised expertise and theoretical perspectives specific to the participating international researchers to undertake parallel analyses of a shared dataset. The use of a shared dataset for the analyses creates a useful point of connection between the various perspectives. This colloquium will report the findings from some of the analyses conducted.

The first presentation explores possible connections between the proportion of time spent on different negotiative foci and students' written product. The second presentation presents an analytical tool for investigating the different types of dialogic talk between students during collaborative problem solving. The third presentation focuses on the affective aspects of the student interactions during problem solving in terms of motivating desires and the influence of the choice of units of analysis on the interpretations of the student exchange. In combination, the parallel analyses presented shed light on the complexity of collaborative mathematical problem solving and learning as social phenomena in classroom settings.

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 223). Umeå, Sweden: PME.

ENTANGLED MODES OF SOCIAL INTERACTION IN STUDENT COLLABORATIVE PROBLEM SOLVING IN MATHEMATICS: CONNECTING PROCESS AND PRODUCT

Man Ching Esther Chan, May Ee Vivien Wan, and David Clarke

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The Social Unit of Learning project examines individual, dyadic, small group problem solving in mathematics in order to investigate the social nature of learning. Utilising a laboratory classroom equipped with 10 built-in video cameras and more than 15 audio inputs, multiple forms of data were collected, including student written products and high definition video and audio recording of every student and the teacher in the classroom. This paper reports the analysis of a class of Year 7 students' social interactions during pair collaborative problem solving of a mathematical task. The possible connections between the proportion of time spent on different negotiative foci and students' written product is discussed.

BACKGROUND

Learning in social settings such as the classroom involves complex processes that require research designs and analytical techniques sensitive to its multifaceted nature and open to analysis from different theoretical perspectives (Clarke, 2001; Clarke et al., 2012). High capacity new technologies and multiple theory-testing can help researchers to investigate the complex processes involved in learning in various social settings. The Social Unit of Learning project investigates the social interactions in learning through a research design that focuses on collaborative problem solving in mathematics utilising the latest available research facilities in Australia.

We propose that student socially performed negotiative activities constitute both an essential aspect of the learning process and a key learning product on which more sophisticated intellectual activity is dependent. In order to investigate this proposition, we have employed student collaborative problem solving as a suitable activity by which the negotiative aspects of mathematics learning can be made more visible. In the Social Unit of Learning project, intact classes of 7th grade students (12 to 13 years old) with their usual mathematics teacher were filmed in a laboratory classroom completing a sequence of mathematics tasks individually, in pairs and in small groups. The laboratory classroom is equipped with 10 built-in video cameras and up to 32 audio channels. The resulting data collected in the project include: all written material produced by the students; instructional material used by the teacher; video footage of all of the students during the session; video footage of the teacher tracked throughout the session; transcripts of teacher and student speech; and pre- and post-lesson teacher interviews. Several parallel analyses have been undertaken to analyse the project data,

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 225-232). Umeå, Sweden: PME.

for example, the student's patterns of dialogic talk, and the affective enablers and disablers of student engagement. This paper reports on an analysis that extends the work by Chan and Clarke (2017b) on the negotiative foci of the student interaction during collaborative problem solving.

THE ENTANGLED MODES OF SOCIAL INTERACTION

The process of learning can be seen as the construction of knowledge by students through their interaction and participation within the classroom setting (García-Carrión, & Díez-Palomar, 2015; Sfard, Forman, & Kieran, 2001). Drawing from the work of Yackel and Cobb (1996) and Brousseau (1986) and analysing the video data generated from the Social Unit of Learning project, Chan and Clarke (2017b) examined the foci of student social interactions during collaborative problem solving. The students were found to engage in negotiations that relate to the mathematical aspects of the task (e.g., the calculation of the size of each room in an apartment) (Mathematical focus) but also about what constitutes an acceptable approach to solving a task (e.g., the scale of the floor plan drawing) (Sociomathematical focus). Students also had to contend with other aspects of the group work setting, such as whether someone is on task or is fulfilling their social obligations within the group (Social focus).

Chan and Clarke (2017b) hypothesised that all three foci of interactions (Mathematical, Sociomathematical, and other Social) co-exist in an entangled form and each represents one avenue to improved learning outcomes in our mathematics classrooms. The delineation of the negotiative foci of students collaboratively attempting problem solving tasks was suggested to alert teachers to instructional contingencies which might be manipulated to optimise student mathematical activity and consequent learning. The researchers suggested the need for further analysis to delineate each of the three negotiative foci and investigating the nature and consequences of their interaction. The current study applies the negotiative foci distinguished by Chan and Clarke (2017b) to analyse the student-student interactions during the pair task attempted by a single intact class of Year 7 students addressing the research questions: How do the student pairs differ in terms of the proportion of time spent on different negotiative foci during collaborative problem solving in the project? And what are the possible connections between the proportion of time spent on different negotiative foci and the students' written solution?

THIS STUDY

The data were drawn from the Social Unit of Learning project which was conducted in a laboratory classroom situated within the Melbourne Graduate School of Education at the University of Melbourne, Australia. One class of Year 7 students (26 students) provide the focus for this report. The class participated in a 60-minute session in the laboratory classroom involving three separate problem solving tasks that required them to produce written solutions. The students attempted the first task individually (10 minutes), the second task in pairs (15 minutes), and the third task in groups of four to six students (20 minutes). The problem solving tasks used in the project were drawn from previous research (e.g., Sullivan & Clarke, 1991).

> Task 1 provided students with a graph with no labels or descriptions with the following instructions: "What might this be a graph of? Label your graph appropriately. What information is contained in your graph? Write a paragraph to describe your graph."
>
> Task 2 was specified as follows: "The average age of five people living in a house is 25. One of the five people is a Year 7 student. What are the ages of the other four people and how are the five people in the house related? Write a paragraph explaining your answer."
>
> Task 3 stated that "Fred's apartment has five rooms. The total area is 60 square metres. Draw a plan of Fred's apartment. Label each room, and show the dimensions (length and width) of all rooms."

Transcript coding

The analysis of negotiative foci employed the negotiative event as the unit of analysis for analysing the transcripts. A *negotiative event* is defined as "an utterance sequence constituting a social interaction with a single identifiable purpose" (Clarke, 2001). Each of the pair work transcript was partitioned into negotiative events and then subsequently coded according to the negotiative focus of each event:

- Mathematical (M): a concern with the facts and procedures required for completion of the task
- Sociomathematical (SM): a concern with the didactical norms of the classroom
- Other Social (S): a concern with social obligation, agency, and responsibility and acceptable behaviour within the group, and non-task related speech.

A detailed coding manual was developed to document the coding rules and procedures. To ensure consistency in coding, a single person (second author) carried out all of the transcript coding and partitioning which was then checked by another person in the research team. Any discrepancies were discussed and resolved among the three people in the research team. Out of all of the transcripts of the 13 student pairs (318 negotiative events in total), one negotiative event of 42 seconds duration (0.31%) was deemed "uncodable" due to some inaudible speech.

FINDINGS

Figure 1 shows the proportion of time that each student pair spent on the different negotiate foci during the pair task. There are variations among the pairs in the proportion of time spent on each of the foci. All of the pairs spent majority of the time during the task (>51%) carrying out conversations that were concerned with the facts and procedures required for task completion (M), with one pair (Pair 11) spending 96% of the time on that particular focus. Two pairs of students (Pairs 11 and 12) did not dedicate any time negotiating the didactical norms of the classroom when attempting the task. The rest of the pairs spent 6% to 27% of the time negotiating with a sociomathematical focus (SM). Some pairs spent more than others on the other social focus (S), ranging from 0% to 33%.

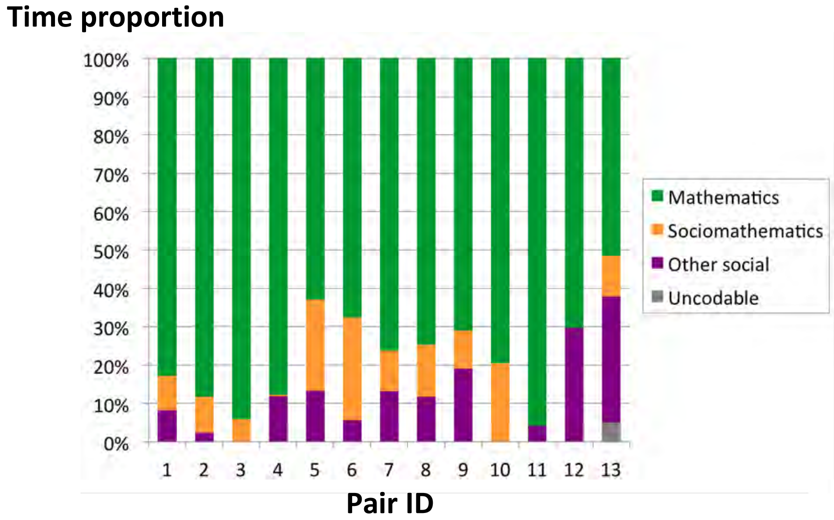

Figure 1: Distribution of student conversation foci during the pair problem solving task (N = 13 pairs).

The distribution of time spent on the different negotiative foci provides a way to summarise and characterise the nature of the student conversations during the pair task. In addition to the video transcripts, the project also collected the students' written solutions to the pair task. The data available allows hypotheses to be made between the process and product of collaborative problem solving. The written solutions of three pairs of students who displayed contrasting negotiative foci were examined. These include Pair 3, Pair 6, and Pair 13 who spent relatively large proportion of time on mathematical focus (94%), sociomathematical focus (27%), and other social focus (33%) respectively. Their written solutions submitted for the pair task are shown in Figures 2, 3, and 4 respectively.

Pair 3 (M: 94%, SM: 6%, S:0% in Figure 1) spent most of their time on negotiating the facts and procedures required for completion of the task but did not culminate in a viable written solution to the problem. As can be seen in Figure 2, the pair did not fill in a final solution in the worksheet provided (left image). From their working out sheet (right image), the pair appeared to understand the concept of average (☐÷5=25), and recognised the task constraint in terms of the age of the Year 7 student (13 years). However, the students appeared to have difficulty coordinating the various mathematical (e.g., average household age) and situational task constraints (e.g., age of Year 7 student and the relationship between the household members):

Ji-na: So we need to get 125.

Nafisa: Oh that's going to be …

Ji-na: The age. So …

Nafisa: Okay. So that - it's simple. Okay. So 25 - I have a very good one. Wait, wait, wait. Twenty-five, 25 equals 50, so one more 25 equals …

Ji-na: So if we put …

Nafisa: Wait, wait, wait. I - I have it, I have it, I have it, I have it, I have it. Yes. That will work, I think. So 28, one, two, three, four, one, two, three, four. So 4, 32, 4, 7, 72. No. That doesn't add up. … why can't they have 25, 25, 25, 25?

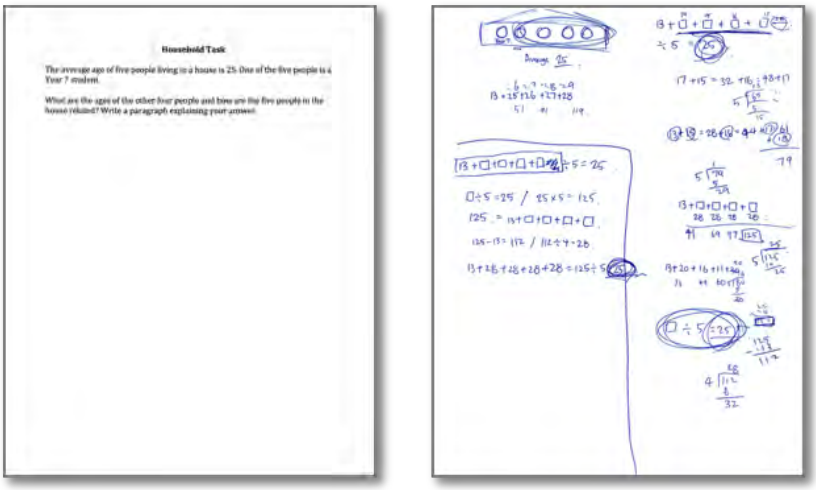

Figure 2: The written solution (left) and working out sheet (right) of Pair 3.

Out of the 13 pairs, Pair 6 spent the most time on Sociomathematical focus (M: 68%, SM: 27%, S: 6% in Figure 1). The written equations (20+40+10+42+13=125 and 125÷5=25) and the list of people and their ages (Figure 3) demonstrated the students' understanding of the task requirements and the mathematical concept of average. The format of the final solution was a result of an explicit negotiation of the solution format (numerical rather than textual) between the students Poya and Pedram:

Poya: You're writing a math equation.

Pedram: Yeah. It's a math equation because it's a maths problem.

Poya: It says write a paragraph.

Pedram: I am going to write a paragraph, you dumb-ass.

Poya: Okay. Well, what – what's going to be in the paragraph? Numbers, equal sign?

The pair therefore explicitly negotiated issues associated with the normative expectation of a mathematics classroom (i.e., the sociomathematical norm) in terms of what constitutes an acceptable solution.

Compared to all of the other pairs, Pair 13 (M: 51%, SM: 11%, S: 33% in Figure 1) spent the least time on the Mathematical focus and relatively more time on the Social focus. Like Pair 3, the pair did not produce a complete solution (Figure 4). Their writing in the working out sheet appeared fragmented and scattered. Their Social focused negotiations focused on discussing each other's obligations:

Ramthu: Come on, let's do it tough, let's work.

Jose: Wait, wait, wait, wait.

Ramthu:	Oh write a paragraph. (pause) I think I get it.
Jose:	Twenty-five.
Ramthu:	… Come on, you're not even working. Hey. You're not working. Jose.
Jose:	Wait. I'm thinking.

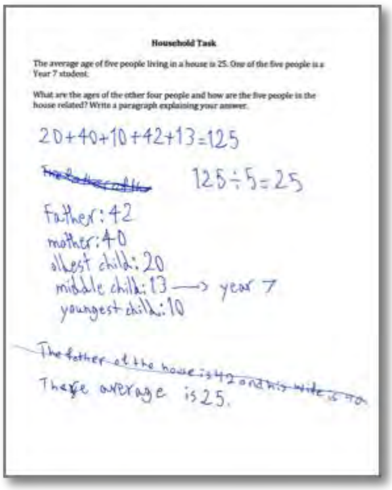

Figure 3: The written solution of Pair 6.

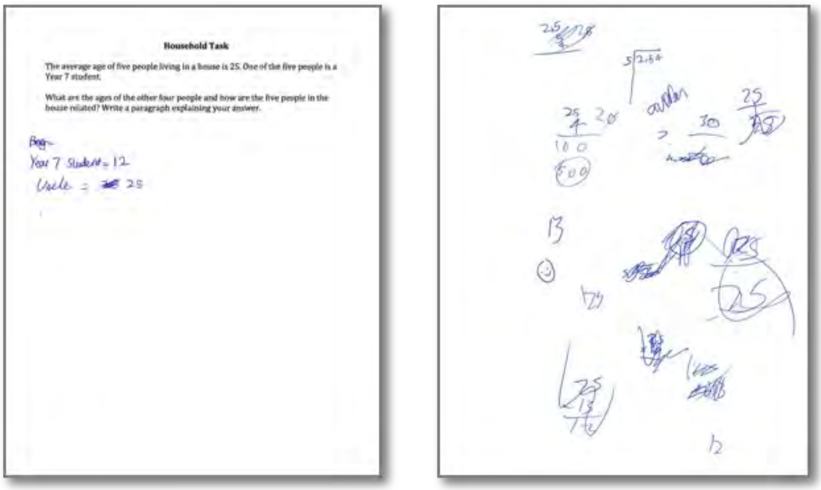

Figure 4: The written solution (left) and working out sheet (right) of Pair 13.

DISCUSSION

This paper addresses two research questions: How do the student pairs differ in terms of the proportion of time spent on different negotiative foci during collaborative problem solving in the project? And what are the possible connections between the proportion of time spent on different negotiative foci and the students' written solution? For the first research question, the 13 pairs of students varied in terms of the proportion of time spent on different negotiative foci during collaborative problem solving. Overall, each student pair spent at least 50% of the time negotiating the facts and procedures associated with the task, which can be seen as an indication of on-task behaviour. However, the presence of the Sociomathematical (SM) and other Social (S) foci also appears to serve an important function in the students' collaborative problem solving process and consequent outcome.

Neither Pair 3 nor Pair 13 produced a viable solution where Pair 3 spent the most time on the Mathematical focus, and the other pair spent the most time on the Social focus out of all of the student pairs. From the transcript and the written products, Pair 3 appeared to be stuck on discussing the mathematical facts and procedures and did not pay a lot of attention to the task context. In their Social-focused conversations, Pair 13 demonstrated concerns regarding each other's obligations. When delving deeper into the transcript, the format of the written solution produced by Pair 6 illustrated a considered negotiation regarding an acceptable solution for a mathematical problem. The explicit discussion of the sociomathematical norm of the classroom appeared useful in addition to their Mathematically and Socially focused discussion.

Based on the above analysis, the findings of the study support the proposition of Chan and Clarke (2017b) in terms of the value of each of the negotiative focus as avenues to improve learning outcomes in our mathematics classrooms. Each of the foci can serve an important function as:

- talk that keeps channels of communication open (social talk),
- talk that keeps the focus on the task in hand (sociomathematical talk, meta-talk), and
- talk that comes from the subject with which the pupils are engaging (mathematical talk, expert talk).

All three must be studied in situ and in relation to each other as they occur in authentic classroom activity. While the presence of some of the talk type may contribute to successful written solution from the students (e.g., Pair 6), the presence or absence of some of the talk type in the student interaction could relate to unsuccessful collaborative problem solving outcome (e.g., Pairs 3 and 13).

It is important to emphasise that the findings presented in the paper constitutes an initial step for establishing the plausible connection between negotiative foci and learning product in collaborative problem solving. Further analysis of the data of the rest of the pairs in the class and that of additional classes is needed to determine whether similar interaction patterns and learning product found in the three pairs are also evident in the interactions of other pairs of students. The analysis in this paper establishes the value of having access to a record of the student collaborative problem solving process and product for cross-referencing and understanding the nature of the student interactions.

In addition, other theoretical perspectives and constructs also need to be examined to consider other possible mediating and moderating variables. The multi-theoretic research design adopted by the project (Chan & Clarke, 2017a) utilises the rich data collected in the study for drawing out and contrasting different theoretical accounts and supporting the generation of more complex explanations of classroom learning.

CONCLUSION

The Social Unit of Learning project involves pioneering work on multiple levels: from the use of the laboratory classroom facility, the focus on social interactions in the context of collaborative problem solving in mathematics, the variation of social units during the investigative session, to the simultaneous recording of multiple group interactions within a simulated classroom setting. With the abundant amount of data that can be generated from the laboratory classroom, data processing needs to be purposeful and strategic to maximise the meaningfulness of the data analysis and the conclusions that can be drawn from the findings. This paper offers a starting point for further theorisation and investigation.

Acknowledgement

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ANALYSING PATTERNS OF STUDENTS' INTERACTION WHEN SOLVING OPEN-ENDED TASKS IN SMALL GROUPS

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In this paper, I draw on previous work (Díez-Palomar & Cabré, 2015) suggesting that not all interactions are equally fruitful in terms of mathematical learning and understanding. I focus on the following research question: How can we use *interactional events* (IE) analysis to understand students learning within small groups? The data of this paper came from the *Social Unit of Learning* project, led by Prof. Clarke at the International Centre for Classroom Research (ICCR) at The University of Melbourne. In order to analyse the data, I created an analytical tool with the support of the ICCR members, to analyse the social interactions among Year 7 students (12-13 years old) as they solve an open-ended mathematical task in small groups (four students per group). The analytical tool is grounded on the approaches of Soler and Flecha (2010), Mercer et al. (2004) and Habermas (1984), to distinguish between dialogical (Type 3) and non-dialogical (Type 2) talk (Díez-Palomar & Cabré, 2015). According to Díez-Palomar and Cabré (2015), students may use either *validity claims* or *power claims* (Habermas, 1984) during group discussions in order to support their arguments, which eventually may be accepted (or rejected) by their peers. Through the analysis of several videotaped IEs of one student group, I found evidence suggesting that the effort that creating the need for students to justify their reasoning based on their validity claims is an effective way for them to learn (or to consolidate their previous learning). The analytical tool used in this study may offer a few pointers for researchers and teachers to distinguish what types of interactions are the ones that would produce deep and meaningful mathematical learning.

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ANALYZING ENGAGEMENT IN MATHEMATICAL COLLABORATION: WHAT CAN WE SAY WITH CONFIDENCE?

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Affective aspects of collaborative learning situations in mathematics can be thought of as a multidimensional and complex system. In this article, Goldin's analytical tool of motivating desires, which is aimed to cover multiple aspects of affect and includes social and contextual elements is examined in detail. An episode of students' mathematical collaboration is analyzed using Goldin's tool. The analysis showed that the interpretations of students' participation and identification are dependent on the choice of analytical unit. Different information is available in different excerpts that precede the analyzed episode. Confidence in the interpretive coding of a given excerpt can be heightened or perturbed by the consideration of preceding episodes.

BACKGROUND

Engagement is an essential element of academic learning, school success and positive future related decisions (see Lawson & Masyn, 2015). The engagement that students have with mathematics learning is related to both affective and behavioral aspects. The studies of engagement are identified to cover emotional and cognitive, but also behavioral elements (e.g., Fredricks, Blumenfeld, & Paris, 2004). Engagement and affect are empirically and theoretically linked with each other, but neither of the two are defined as a subconcept of the other. For example, student's engagement has an affective dimension indicated by how the student values school and what his/her affective belonging is (e.g., Voelkl, 2012).

According to Lawson and Masyn (2015), engagement combines what students will do with what they can do, forming a participation-identification model. Participation is defined by students' behavior (in what ways and how intensively they participate in school activities), whereas identification is an affective variable defined by students' sense of school belonging and their affective valuing. The participation-identification model acknowledges the independence between the elements. Lawson and Masyn (2015) showed that students' behavior is not necessarily connected with their identification. Instead, the participation-identification model sees students' behavioral engagement as functioning differently across the social settings, and being impacted by situational uniqueness and subpopulation difference. As a result, Lawson and Masyn (ibid.) argue that researchers must pay attention to the interpretations of students' behaviors, as behaviors are influenced by students' interpretations of their social situations. Goldin (2017) argues that investigating students' in-the-moment classroom engagement with mathematics requires consideration of several domains dynamically

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influencing each other in complex ways, including cognition and metacognition, motivation, affect and meta-affect, and social interactions with the teacher and with peers. However, many analytical tools focus on specific aspects of affect, and some of them require careful decisions over the excerpt under investigation. The chosen excerpt frames the unit of analysis, which might determine the inferences that can be reliably and confidently drawn.

The motivation behind this article is the realization of this complexity in the course of analyzing students' collaborative work from an affective perspective. Goldin's (2017) tool of motivating desires approaches this challenge through student's immediate motivating desire or goal, which according to him can incorporate aspects of all of these domains. Motivating desires draw from the framework of engagement structures (Goldin, Epstein, Schorr, & Warner, 2011), and they are described as individual needs reflecting what happens and arises in social situations. They are not meant to be classified as either good or bad, and are defined as profoundly affective and existing in collaborative situations in mathematics classes (Goldin, 2017). The article aims to investigate what part of students' affective interaction during mathematical collaboration can be made visible for research purposes in a valid, systematic and reliable way, and how this is dependent on chosen unit of the analysis. The hypothesis is that the choice of the unit of the analysis will have an impact on the interpretive accounts constructed by the researcher. In particular, the investigation addressed the question of whether the units of analysis are independently coherent (disconnected entities) or progressively explanatory (each adding information to others)? In this article, I will apply Goldin and his colleagues' (2011; 2017) analytical frame and follow a procedure to examine at what point and with what additional information would my interpretation of a particular affectively rich 1-minute episode be considered to be stabilized? This stabilization would justify an increase in the confidence in the interpretive coding of the episode.

METHOD

The data used in this study is part of a video record collected within a larger research project called, "The Social Essentials of Learning" (see the introduction of the project in Chan & Clarke, 2017). The research design documents individual, dyadic, small group and whole class problem solving and learning in mathematics. The data used in this study consists of a video excerpt where four students (two girls, two boys, Year 7) participate in a 20-minute (as part of an altogether 60-minute session) researcher-designed and teacher-facilitated session involving an open-ended mathematical task. The task had multiple possible solutions, included symbolic or graphical elements, and afforded connection to contexts outside the classroom. The session was video recorded, all student work was collected and speech was transcribed for analysis.

Goldin and his colleagues (2011, 2017) have strived to make the list of motivating desires as explanatory as possible, but yet they don't claim the model as comprehensive (see the list of labels in Table 1). Some of them require information from the students and are thus not reliably identifiable through the video: for example, it is hard to tell the difference between "Look how smart I am" and "Let me teach you" just by the student's behavior. As Goldin and others (2011) put it, the particular *expressions* of affect can differ substantially according to sociocultural norms and across individuals, and yet the underlying structure can be essentially invariant. To assist the consistency of my interpretations, I grouped Goldin's labels into six categories (Table 1). In the analysis, I made particular use of the descriptions of labels (Goldin et al, 2011; Goldin, 2017) displayed in Table 1.

Categories	**Original labels by Goldin**	**Descriptions**
Commitment	Get The Job Done I'm Really Into This (flow)	The desire to complete an assigned mathematical task or experience the very activity
Outperforming	Look How Smart I Am Check This Out Let Me Teach You I'm Right You're Wrong	The desire to impress others, obtain a reward or "payoff" or to help another understand or solve the problem
Respect Me	Don't Disrespect Me It's Not Fair Acknowledge My Culture	The desire to defend oneself or redress a perceived inequity
Avoidance	Stay Out Of Trouble Pseudo-Engagement Don't Notice Me I Want Out	The desire to avoid interactions that may lead to conflict or to look good to the teacher or to peers by seeming to be engaged
Give Me Your Attention	Help Me Focus On Me Value Me	The desire to get noticed, be cared or get helped
Stop The Class	Stop The Class	The desire to interrupt the ongoing activity of others

Table 1: The framework of motivating desires, their descriptions, and their categorization for the purposes of this study.

Goldin's (2017) tool of motivating desires was originally developed based on *key affective events* in students' collaboration. A key affective event is defined in Schorr and Goldin (2008) as an occasion in the context of doing or discussing mathematics, where significant affect or a significant change in affect of a student is expressed or can be inferred. For the analysis here, I chose a situation wherein three out of four students expressed a visible behavioral or affective change. I started the analysis by labeling all the affective actions and elements in that situation according to Goldin's (2017) tool of motivating desires. After that, I investigated the previous 30 seconds aiming to find out whether the consideration of new information would affect my coding of the original 1-minute episode. I utilized my analysis of the preceding 30-second episode to elaborate or amend my interpretations of the original episode. I then repeated once more the backtracking process, using the preceding 30-second episode (prior to the one just analyzed) and revised my interpretation of the original episode accordingly.

RESULTS

The video excerpt starts at 44.30 minutes from the beginning of the whole session, where the students have already worked through one individual task, one pair task and have had nearly fifteen minutes to work on the group task. They had been given all the instructions regarding the group work by the teacher, read the task, discussed it and started to work on it. Their task was to design and draw an apartment of 60 square meters with five rooms, and to come up with a single solution within the group. The students were seated at a triangular-shaped table in pairs: two girls were on one side, two boys were on the other side, while the third side was empty: this way all the students' faces were visible. From the camera's perspective, the students from left to right were Anna (girl), Pandit (girl), John (boy), and Arman (boy).

Episode 0, one minute starting point: 44.30-44.30

During the episode all students seem to work on with the task. Girls do pair work: they discuss their ideas and outline them into their working sheets. Boys seem to be individually pondering the task, not making much visible interaction. There is no interaction among the whole group in this original episode. Neither of the boys talk during this minute. Anna is faced towards her working sheet throughout the whole period. Pandit is the only one who looks at each member of the group at least once. John appears to be quite concentrated on the task, Arman is staring at nothing in particular.

In this episode, Anna's category of motivating desires is *Commitment*. Anna's orientation stays similar during the whole episode. She works on the task, is physically oriented towards her worksheet, and talks sparingly, addressing Pandit only. Anna appears to be very engaged with the task. She *participates* working on the task with Pandit. However, Anna doesn't put effort into making the whole group work together – it does not look like if she was having a strong *identification* with the group or *participation* with the collaborative assignment.

Pandit's motivating desires can be considered as *Commitment* and *Give Me Your Attention.* At the beginning of this episode, she is faced towards Anna and the worksheets. She smiles, looking excited. Then she turns back to her own sheet, still smiling, and starts working. She leans towards the table and doodles a sketch of the apartment on her sheet, talking aloud. She makes a mathematical argument to Anna ("…You've got one here, there's three on three… 44.48), then the girls discuss their ideas and both keep working on their separate worksheets. She suddenly addresses boys: "Guys, how big do you want the living room to be?" (45.07), the boys don't answer her - they don't seem to notice her - she looks at the boys once, a bit confused (45.08-45.14). She turns back towards Anna, and they continue working, smiling and relaxed. Pandit is engaged in both ways. She *participates* and she is also *identified* with the situation: she works with the task, makes an effort to make the whole group work together by asking the boys' opinion, and shows emotions. She seems amused by the situation, and thus it seems that she is *identified* with the social activity in addition to the cognitive activity.

In this episode, John's motivating desires can be categorized as *Commitment, Give Me Your Attention* and *Avoidance.* John starts the episode pondering the task, arm around his neck, staring fixedly at nothing in particular (44.30-44.37). He drops his ruler and bends down catching it, then comes back sitting (still silent) and pondering (44.40-44.49). He seems to become a bit frustrated with the task (looks at the task, shakes his head, fiddles with the ruler) and turns his head away (44.49-45.00). He turns his head back towards the task and keeps on pondering and fiddling the ruler, then he bends a bit more towards the task, still pondering, and finally moves his position once more. He looks at his sheet of paper all this time (45.00-45.30). John seems to be engaged with the task, even though his *participation* seems quite passive. He thinks instead of talking or asking or making efforts to the worksheet, and does not express positive emotions. However, his *identification* with the cognitive activity seems strong, he wants to work on the task. On the other hand, he does not *participate* in the group work, and his identification with the social situation seems low.

Arman's categories of motivating desires in this episode are *Commitment* and *Avoidance*. Arman is bending down to pick up his fallen ruler (44.30-44.39) at the beginning of this episode. His face remains turned away from camera: he might be pondering, but also disengaged, as he fiddles with the ruler and looks away from the sheet and the table (44.39-44.56). He moves his chair so that his face is visible again; he fiddles with the ruler and a pen, not looking at the worksheet (44.56-45.02). He suddenly stands up and seems to get interested in the task: he looks at the girls' working sheets but doesn't say anything (45.02-45.07). He turns away still standing, and sits down. He does not react to Pandit's question (45.07-45.14). For the rest of the episode, he looks at his and John's working sheets and fiddles with his ruler and stays silent and passive (45.14-45.30). Arman seems partly engaged and partly avoiding. He does not make any clear efforts either in working with the task, nor in making group work. His *participation* is minimal, and his *identification* with the collaborative assignment seems shallow.

Episode -1. The previous half a minute: 44.00-44.30

The new information in this episode is that Anna shows participation in the social interaction (To Pandit: "I don't think they like you as well" as an opinion to a statement Pandit makes about the boys, 44.20). Still, her only extrinsic signal towards social participation is her statement. Otherwise she keeps working with the task, not looking at the others at any point. Pandit gestures and calls for the guys ("Guys guys what are you doing?" 44.00), gets an unclear utterance (audible, but vague) from John and looks a bit uncomfortable, then she gives a mathematical answer to John ("It's not the same area" 44.06). She gets an unclear answer from John, gets confused, laughs and nestles down, pressing her face between her arms, towards the table (44.08-44.15). She straightens herself, seems emotional (smiling and confused), turns back to the task and Anna (44.15-44.30). John says something unclearly to Pandit (44.04), then again smiling and trying to look confident; he nods his head repeatedly (44.08-44.12). John has problems with his language. It looks like he notices that his communication with Pandit has failed: he sees Pandit bending down, he quiets down, takes his ruler and starts drumming with it seemingly nervously, and he says "aaooh" and hits the ruler to the table intensively. Then he smiles, turns towards Arman, and starts to fight with him with the ruler (44.12-44.22). After that, John calms down and sits up staring at his sheet. He is clearly participating in the social interaction though not task related, probably due to his position as a peer with language problems. It is hard to identify a suitable motivating desire here. Avoidance would probably fit best: Pandit and John avoid eye contact with each other for a while after John's failed attempt at communication. In this excerpt, Arman just moves around in his chair. It is clear now that his motivating desire is Avoidance. He stares at nothing, and he barely reacts when John fights with the ruler towards him. *Additional Insights:* Episode -1 clarifies Arman's motivating desires. It opens up Anna's subtle interaction with what happens socially, indicating social identification in addition to cognitive engagement. It also brings up John's language problems and a socio-cognitive challenge the group needs to solve concerning how to collaborate. Pandit seems to be the only one addressing this challenge, as she makes a gesture towards the boys in the original episode. Thus, this episode strengthens the interpretation of Pandit being identified both with cognitive task and the social challenge of group work. John's language problems are not in contradiction with the previous interpretation of him being engaged with pondering the task, but the social challenge revealed in this episode shifts the interpretation of his later focusing on the task from task engagement to a possible tool to avoid social interaction.

Episode -2. The previous half a minute: 43.30-44.00

In this episode, Anna shows even more affective expressions. Anna was smiling, she appeared to be open to the whole group, and she jokes with Pandit about their working ("Oh yeah, good on you" (43.42). She shows clear identification with the social interaction now. She even talks to the group, even though she is not directly addressing the boys. Pandit shows all the same features of engagement as she showed in the other episodes. She is emotional (smiles, looks excited), she jokes ("At least I drew it lightly" 43.36), she acknowledges each group member, she participates in the mathematical discussion in an outperforming style ("It's common sense, you need a kitchen"43.48), and she tries to involve the boys, partly indirectly ("What are they doing?" 43.58 – "Guys, what are you doing?" 44.00). John laughs a bit with Arman at the beginning of this episode, and then he turns back to his sheet and keeps working with that. He also asks for the rubber from the girls (43.56). His behavior strengthens the earlier interpretation of him being engaged with the task, and also that he is not afraid to be in contact with the others. He seemed to be willing to strengthen his identification with the group. Arman acted similarly in this episode compared to the others – the original interpretation of his affective behavior remains unchanged. *Additional Insights:* Consideration of Episode -2 consolidates rather than challenges the earlier interpretations based on the original episode and consideration of Episode -1.

CONCLUSIONS

The purpose of the paper is to highlight the interpretive nature of a well-structured analytical schema by Goldin and his colleagues (2011; 2017) in terms of the question: What can be said with confidence about students' affective engagement structures during mathematical collaboration? The first interpretation based on identifying the students' motivating desires in the original episode would have been too narrow regarding Anna and John, and possibly invalid regarding Arman, whose behavior in the original episode suggested Avoidance instead of Commitment. Also the interpretation of the affective engagement of the whole group seems likely to have been mistaken if it was based only on the original episode. In that episode, it looked as though the students' were just working in pairs (girls) or individually (boys). The students' efforts towards making the group work together, and especially the challenges to do so were revealed in the earlier episodes. Thus, the interpretation of a chosen episode is greatly strengthened by consideration of the additional information provided in earlier episodes.

From the identification-participation model's point of view (Lawson & Masyn, 2015), the study showed interesting interplay especially visible in John, who seems to play two different games. He wants to fulfill the requirements of the task and the learning expectations, and seems both participating and identifying with that goal. He also strives to find a way to fit socially and culturally to the interaction by making varying efforts and showing persistence in the face of communication difficulties. He displays several affective expressions from an emotional moan to nervous drumming and confused smiling. This suggests the engagement process during collaborative work sometimes creates discrepancies between student's self-belief and the social expectations of the situation, which is in line with Goldin and others' (2011) statement of emotions being often expressive of interactions between the individual and the social.

In Goldin and others' study (2011), it was stated that while trait variables importantly influence behavior, the immediately construed situation has still greater significance. This study's results are in line with that argument. The tool of motivating desires proved to be a useful basis to analyze the affectively rich moments in this study.

However, careful attention must be paid when defining the analytical unit through which to apply it. Judgements regarding the validity of the interpretive accounts will depend on how the operationalized affective components accommodate or ignore the contextual and historical background of the episode and whether a systematical analytical approach can make use of such background information. This paper has attempted to illustrate how such an analytical approach can be applied. In the future, motivating desires' relationship between students' behavior will need more elaboration. It is also necessary to gain more knowledge about how a particular motivating desire connects with students' participation and identification.

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COLLOQUIUM 2

FOREGROUNDING DAVYVOD'S CURRICULUM: RELATIONAL APPROACH AND ALGEBRAIC THINKING IN EARLY GRADES

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Based on a dialectical materialist theoretical understanding of humans and the world, Vasily Davydov (2008) articulated an educational program known as the Elkonin-Davydov curriculum that features a theory of developmental instruction. Two of the chief ideas of this theory are (1) a relational approach to number concept —which entails shifting teaching and learning from arithmetic operations to quantitative relationships—and (2) a materialist view about theoretical thought —which emerges in practical, culturally shaped activities with artefacts, language, and gestures. In this colloquium, two research reports and a theoretical essay explore opportunities in developing algebraic thinking that arise from Davydov's work. The authors frame knowledge development as an increasingly sophisticated, artefact-mediated, dialectical articulation of actions, gestures, signs, and words (Radford, 2011). Mellone et al. (research 1) implemented the Davydov curriculum in Grade one, and analysed the attention paid by the students to the structure of the situation (pouring liquid from two containers is put into one) and to the relationships between quantities. The students expressed the additive relationship using graphical representations, surfaces of rectangles, and simple algebraic expressions. Ericsson (research 2) used a context of mixed fractions to facilitate students' development of algebraic thinking. This paper suggests that the mixed fraction encodes two relationships: an additive relationship between the integer part and the fractional part and a multiplicative relationship between the numerator and the denominator of the fractional part. Polotskaia et al. (theoretical essay) critically review the epistemology and existing classifications of multiplicative situations. Based on Davydov's relational perspective, this paper proposes a new classification of multiplicative structures and a set of graphical representations to implement in teaching and learning contexts aiming relational and algebraic reasoning development. All three reports emphasise the relational approach to teaching mathematics and the special role that graphical and other semiotic devices play in the emergence of culturally shaped theoretical knowledge.

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ALGEBRAIC THINKING AMONG GRAPHICAL REPRESENTATIONS AND ALGEBRAIC SYMBOLS

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Based on evidence showing the presence of Algebraic skills in very young children, we will discuss some theoretical aspects connected to Algebraic thinking and its expression by means of graphical representation. In doing this, we also refer to an ancient example of manipulations of surfaces of rectangles in the resolution of second-degree problems of the Babylonian period. Subsequently, we will present an episode that occurred during an experimental path inspired by Davydov's Curriculum developed in a first-grade class. In this class, we have actually observed that the activities rooted in the observations and work on real quantities give rise to pupils' Algebraic thinking, expressed through the coordination of verbal language, actions, graphical representations and Algebraic symbols.

INTRODUCTION

In recent years, many studies have highlighted the presence of naive forms of Algebraic thought in very young children (for an interesting survey on the topic, see Kieran, Pang, Schifter, & Fong-Ng, 2016). In one of these studies, as an example, the authors advanced a very interesting hypothesis:

> It is possible that these days algebra is simply "in the air": elements of algebraic discourse may be present in other school discourse well before its formal introduction in 7th grade. With help from the media, algebraic forms of expression may even be infiltrating colloquial discourses. (Caspi & Sfard, 2010, p. 256)

Beyond what may be the cause, the presence of early Algebraic skills in pupils has challenged the "traditional" didactical path from Arithmetic to Algebra, opening up new scenarios for mathematics education. Indeed, we think that no educational strategy should be planned without fostering these early skills; among the goals of primary mathematics education, we should include the need to refine these early forms of competence, acknowledging and appreciating them early and driving their development towards the use of the formal Algebraic language. Actually, we should add that the didactical proposal to start from Arithmetic to only later in the lower secondary levels propose Algebra is "traditional" only for some countries, in particular for the Western ones. Indeed, a new awareness about different approaches for K-6 mathematics education practices comes from studies and surveys like the one presented by Cai and Knuth (2011). In this book, the authors point out that one of the features

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 247-254). Umeå, Sweden: PME.

emerging from the math curriculum of primary schools in Eastern Countries (like China, Singapore, but also in Russia concerning Davydov's curriculum) is a particular focus on Early Algebra, essentially rooted in a different perspective of the didactic of Arithmetic. In particular, these mathematics curricula explicitly state that their main objective in teaching Algebraic concepts is to deepen students' understanding of relationships between quantities through extensive use of graphical models (Cai & Knuth, 2011).

In this regard, after a theoretical framework in which we will present our vision of Algebraic thinking and its possible development through graphical representation, we will present the implementation of one context of teaching mediation inspired by Davydov's curriculum and designed to stimulate the rise of Algebraic reasoning in first-grade pupils. We will see how some of these pupils' arguments arise from the effort to coordinate the manipulation of real quantities, verbal language, Algebraic letters and graphical representations.

THEORETICAL FRAMEWORK

Our approach can be placed within the mathematics education research stream that tries to catch the complexity of individual thinking as materialized in the body (by means of actions, gestures, facial expressions, eye movements, etc.) and in the use of signs and artefacts. In this view, as opposed to some cognitive approaches, thinking is not considered something that just happens 'in the head', but rather it is conceived as a social practice rooted in bodily movements and in the use of signs and artefacts (Radford, 2011). In accordance with the previous perspective, the analysis of pupils' behaviours relies on the notion of semiotic nodes, "pieces of the students' semiotic activity where action, gesture, and word work together to achieve knowledge objectification" (Radford, Demers, Guzman, & Cerrulli, 2003, p. 56). Indeed, in this scenario the learning is defined as a process of objectification:

> Learning consists of positioning oneself reflectively and critically in historical forms of action and thinking. Functionally speaking, learning is conceptualized in terms of processes of objectification—i.e., activity bound social processes through which the students encounter and grasp the historically-constituted forms of action and thinking. (Radford, 2010, p. 73)

In particular, in this paper we deal with Algebraic Thinking, by referring to the idea of the "attention to structure" and relationships:

> […] learners exhibit this type of attention when they begin to focus on what stays and what changes – that is, "becom[e] accustomed to considering invariance in the midst of change." (Mason, Stephens, & Watson, 2009, p. 13)

According to this idea, and using Radford's perspective, we defined Algebraic thinking as the kind of thinking that is expressed in bodily movements (gesture, actions on objects, etc.) and use of signs (verbal language, Algebraic letters, graphical representations, etc.) in which we can recognize this *attention to structure* (Mellone & Tortora, 2017).

By this view, we see the Algebraic thinking through different humans' expressions and manifestation, and in particular, we argue that before the acquisition of formal Algebraic language, the Algebraic thinking is mostly expressed through the use of verbal language and graphical representations. These intermediate Algebraic expression ways/registers/modes seem to have been traversed historically in the development of mathematical thought. Indeed, some archaeological finds from the Old Babylonian period, so many centuries before the rise of the specific literal language of Algebra, show very sophisticated forms of human reasoning about solutions of problems, nowadays solvable with second degree equations, built on the drawing and manipulations of rectangle surfaces (Radford & Guerette, 2000). The historian Jens Høyrup described these solution methods with the expression "Naïve Geometry" (Høyrup, 1990). For example, the simplest of these geometric problems regards the research of the length of the side of a square, knowing that the sum of the unknown quantity and the area of the square is equal to ¾ (Høyrup, 1990). This problem in the modern Algebraic language would take the form of the Algebraic equation:

$$x^2 + x = \frac{3}{4},$$

It was solved by Naïve Geometry through some verbally described graphical operations represented by the succession of images in Fig. 1. So, it seems that historically, forms of Algebraic thinking can be recognized in the manipulations of graphical representations, in particular of quantities of areas.

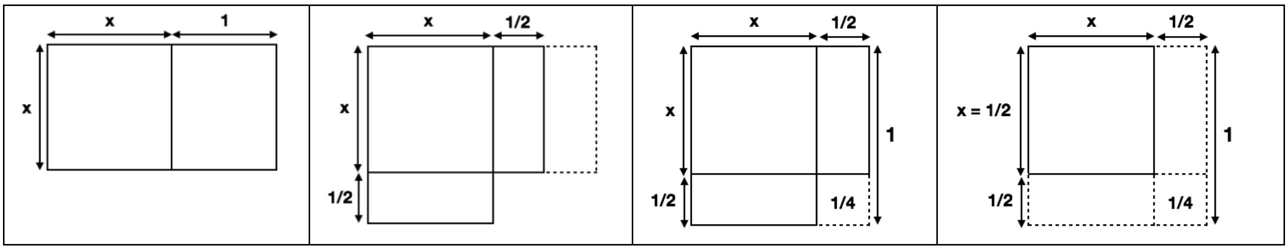


Figure 1: Four-step Naïve Geometry solution of the problem (Høyrup, 1990, p. 450). To be noted that the length of the square side is represented as an area of a rectangle with a dimension equal to 1 and the other one equal to that of the square side.

The notion of quantity and, in particular, the potentialities of experiencing manipulation of quantities to introduce Algebraic language to pupils, is the core of the math curriculum designed by the Russian psychologist Davydov (1982). In these studies, the thesis is advanced that at the root of the Algebraic structures, and of the concept of number, there is the notion of physical quantities. From this thesis, Davydov derives the necessity of letting pupils work with concrete quantities since the very beginning of the school path and rightly before working with numbers. In the context of these manipulative activities, pupils are invited to discover end register, by means of Algebraic symbols, for example, relationships between quantities of liquids or block constructions. In this perspective, the additive structure is rooted in the experience of comparing two different quantities and of trying to make them equal by means of one of two actions: either to decrease the bigger one (subtraction) or to increase the smaller one (addition). In this situation, the crucial point is that in both cases, the quantity to be added or subtracted is the difference between the two initial quantities. To support very young pupils in really understanding these relationships, the role of bodily handling is gradually reduced, and the scholar suggests using graphical representations, for example, juxtaposed segments, in order to better visualize and manage the relationships between quantities.

Our proposal of combining Davydov's ideas with Radford's theory of objectification arose from the hypothesis that the activities proposed by Davydov can support pupils in the process of objectification of Algebraic language. In other words, we believe that the exploration of suitable contexts involving continuous quantities can support pupils making their own sense of the Algebraic language that embodies historical forms of thinking (Radford, 2010). Indeed, the development of a proper use of Algebraic language by students is a very long and complex process, with different levels of linguistic competence. In this perspective, we have found powerful for our study the notion of algebraic "babbling" (Malara & Navarra, 2003), a metaphor to describe the first naïve use of algebraic language by pupils, when improper uses or errors frequently occur.

AN EXAMPLE OF EARLY ALGEBRAIC THINKING

We intend to illustrate some findings from a didactical path inspired by the work of Davydov (1982), which was carried out in a first-grade class in Italy, during the entire school year by one of the authors who was the teacher of the class. All the proposed activities gave great emphasis to the manipulation of concrete quantities and were aimed at recognition and representation of order and Arithmetic relationships through the Algebraic symbols. The path represented the class's first approach to the additive structure. During the activities, there were alternating individual work phases and collective discussion phases, around several problems posed by the teacher. The tasks concerned volumes of liquids in various containers or lengths of ribbons and different manipulative operations on them. Here, due to space constraints, we present just some excerpts coming from two different times of this experimental path with the pupils' fascinating discoveries and productions.

The first activities of the path started from the observation and the discussion over three real identical, unscaled containers, posed on the desk and filled with volumes of water. The teacher suggested referring to the three volumes of water with the letters A, B, and C and she showed to the children the Algebraic symbols for "major of" and "minor of." In Fig. 2, we can see a pupil's protocol where we can see inequalities and equalities arisen from the observation of the three volumes of water, written in symbolic language and a graphical representation in which the containers are sketched as three equal rectangles, and inside them, the volumes of water correspond to areas of blue rectangles. It is interesting to see how the pupil decided to write the equality between the volumes of water A and B two times, not taking for granted the symmetric property of the equality relationship.

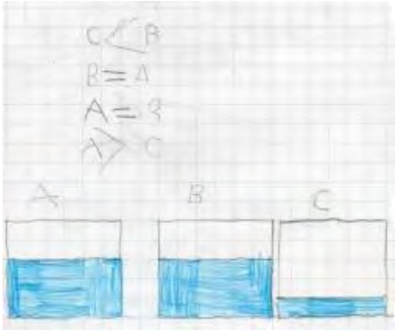


Figure 2: From the notebook of a pupil.

At later time, the context of the volumes of liquids was also used to introduce the additive structure. In particular, the operation of addition was at first proposed to the class through the actual operation of pouring the liquid (cola) of one container into another. This kind of manipulation, thanks to the teacher mediation, led during subsequent activities to the meaningful intuition of the idea of sum, shown by the following collective discussion extract, in which A and B represented two initial quantities of water and C the union of them, obtained by putting the B into the glass in which initially there is only the A:

Davide: “It takes a long time for C to appear, because when B goes in A, then C appears.”

Teacher: “If I take B and put it back in its place, C is not there anymore?”

Domenico: “No, there is always C; C there is also when B is not inside A!”

Teacher: “And where is? Could you show me?”

Giovanni: “It is hidden; it is inside both of them. There is, but it is divided.”

Teacher: “How could we indicate it?”

Claudia: “If Domenico says C is inside them both, he can put the two glasses close and join them!”

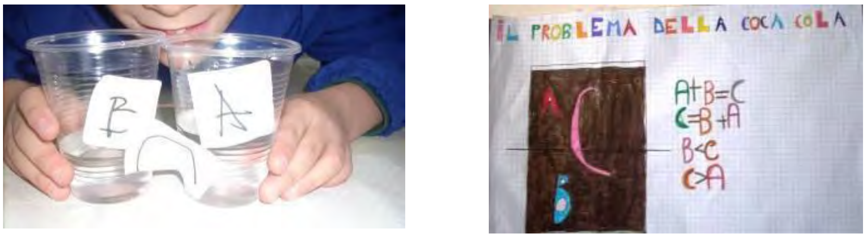
[Domenico arranged the two glasses with two papers with the letters A and B closer, then he put the letter C between them, see Fig. 3a.]

Masha: “It can be called in two ways: C and also A and B.”

The dialogue firstly shows a very interesting perception of a physical transformation: Davide said: “it takes a long time for C to appear.” In general, a transformation suggests a use of the letters as a functional variable, like A(t)=C, in which A is transformed into C in that time. Then the teacher proposed to imagine the inverse action of taking away the B quantity from the glass containing A and invited pupils to reflect on the C existence, regardless of the pouring physical transformation. This mediation favoured the emergence of a ideational meaning of C, and consequently a use of the letters as Algebraic constants. This phenomenon is shown by the interventions of Domenico (“No, there is always C; C there is also when B is not inside A!”) and Giovanni (“It is hidden; it is inside both of them. There is, but it is divided.”). Moreover, the pupils started to recognize the relationships between the three quantities A, B and C, by focusing on what doesn’t change (A and B) before and after the transformation of putting them together. After Giovanni's intervention the teacher invited the pupils to indicate the quantity C after removing the B from the glass of A. In this way she led Claudia to imagine a new action performed by Domenico with an unexpected, wonderful conclusion: to put the C, written on the paper, to join the two glasses containing the A and the B (Fig. 3a). This action on the objects suggests a new Masha's intervention and her words ("It can be called in two ways: C and also A and B") seem ready to be translated in the Algebraic expression C=A+B.

Indeed, the above dialogue was followed by a collective realization of a poster representing the previously experienced cola pouring operation and the discovered relationships by means of the Algebraic language (Fig. 3b). The graphical representation chosen for the poster shows a brown rectangle area representing the volume of cola, inside it a beautiful example of algebraic babbling occurred (Malara & Navarra, 2003): the letter C is written bigger than letters A and B, almost to keep in the size of the letters the order relationships between the three quantities.

This poster, in which the graphical representation with the letters of different sizes inside the brown area posed beside the Algebraic expressions, is a beautiful example of a semiotic node. Actually, the dialogue—together with Domenico's action on the glasses (Fig. 3a) and the poster (Fig. 3b)—represent an example of choral *semiotic node*, where the words, the actions on the objects, and the representations produced by this group of pupils chase each other in the effort to grasp and crystalize the transformations on real volumes of liquids in Algebraic relationships.



(a) (b)

Figure 3: a) Domenico's creation. b) "The Coca Cola problem" poster produced by the class.

CONCLUSION

We described Algebraic thinking as a form of thinking expressed by different human's expressions and manifestations in which we can recognize a certain attention to structure and relationship. In particular, we argued that before the acquisition of formal Algebraic language, the Algebraic thinking is mostly expressed by means of the use of verbal language and graphical representations. In this scenario, we also refer to a historical example, in Babylonia many centuries before the rise of the specific literal language of Algebra, in which sophisticated Algebraic reasonings were accomplished by the manipulation of rectangle areas.

Coming back to our time, we believe that "algebraic thinking does not appear in ontogeny by chance, nor does it appear as the necessary consequence of cognitive maturation. To make algebraic thinking appear some pedagogical conditions need to be created" (Radford, 2010, p. 79). Indeed, the development of Algebraic thinking requires the development of a particular way of thinking such as: analyzing relationships between quantities, observing structures, studying changes, generalizing, solving problems, modeling, making predictions, arguing and proving (Cai & Knuth, 2011). In particular, according to with Davydov (1982), we believe that the context of the relationship between quantities contributes to the development of Algebraic thought both before and after having explored numbers and Arithmetic calculations (Mellone & Tortora, 2017).

In this study, we have presented two little excerpts from a path for the first grade, inspired by Davydov's curriculum. The latter was organized along tasks concerned with the observation and reflections on the relationships between different volumes of liquids in various containers (and lengths of ribbons), and different manipulative operations on them. In this context, the teacher introduced the symbolic language of Algebra in order to help pupils express the relationships between the quantities before and after some transformations. This suitable mediation, which represents for the pupils their first meeting with the Algebraic language, allows them to build their meaning of the symbols, in an educational context where the physical quantities were always the referents of the speeches. In our opinion, the introductions of the Algebraic language in the context of work with concrete quantities helps the occurrence of semiotic nodes between words, graphical representations such as rectangular areas, and Algebraic language. This study, together with previous research (see for example Mellone & Tortora, 2017 and Kieran *et al.*, 2016), shows that this seems to be an important choice in order to create suitable educational environments to make Algebraic thinking potentially arise.

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IDENTIFYING ALGEBRAIC REASONING ABOUT FRACTIONS

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The issue for this paper is to identify algebraic reasoning through students′ sense-making actions, during a lesson, where students and a teacher develop learning models for mixed numbers. The analysis focuses the students' work, trying to make sense of the unknown fractional part of the number. This unknown part was elaborated when the students suggested to "add a little bit more" to construct equality. The unknown part developed to a fractional part with help of an emerging learning model containing algebraic symbols: B=W+p/a. In this activity, I identified potentialities in the students' algebraic reasoning; an additive relationship between the integer and the fractional part of the number, and a multiplicative relationship between the numerator and the denominator in this fractional part.

INTRODUCTION

There is an assumption that mathematical education for the youngest students can be organized in relation to an algebraic tradition described by van Oers (2001). In this tradition, problem solving and algebraic reasoning are essential for the development of mathematical thinking. Davydov's (2008) suggests that an algebraic tradition can be designed through the theoretical framework–learning activitydeveloped to emerge students' theoretical thoughts, in object-related processes, in a practical, cultural–historical activity. In the research field of early algebraization, Davydov's ideas are frequently referred to as means to develop algebraic reasoning, jointly with young students (Cai & Knuth, 2011). In this research field, the underlying purpose of developing algebraic reasoning is to deepen children's understanding of the structural forms, relations, and generalities in mathematics (Cai & Knuth, 2011). However, certain questions in this research field still require answers (Blanton & Kaput, 2011; Nunes, Bryant, & Watson, 2009; Lee & Hackenberg, 2013). Considering their questions, I discuss what can be identified as an emerging algebraic reasoning about fractions: What indicates algebraic reasoning, identified through students' sense-making actions, when the students are analysing quantities?

THEORETICAL BACKGROUND

The concept of objectification

In the algebraic tradition, learning is seen as object-related processes, as processes of objectification, by which students gradually become acquainted with historically con-

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stituted cultural meanings and forms of reasoning and actions (Davydov, 2008; Radford, 2015). These processes are rooted in a dialectical materialistic conception of humans and the world (Leontiev, 1978; Radford, 2015). Following on that, objectification posits an object and the thought about the object as heterogeneous entities (Radford, 2015). In the process of objectification, an object of knowledge can be seen as opportunities for actions, for example, opportunities to calculate. This means that objects of knowledge cannot be accessed directly, they are always mediated by activity, by sense-making actions. Furthermore, according to Radford (2015), the conception of objectification is based on a distinction between two related categories in an object of knowledge: the potentiality and the actuality. Potentiality can be understood as the capacity, ability, or power to do something. Objects of knowledge belonging to this category are treated as undeveloped, lacking in connections with other things, poor in content, and formal in that the object is not yet connected to any concrete examples. Actuality, on the other hand, is described as "being-at-work", i.e., that something in motion occurring in front of us or through our work. Between the potentiality and the actuality are sense-making actions. So, knowledge and its objects have three elements: their potential, their actualization, and the sens-making actions in the activity that mediates them. A motion of an object can appear through the contradictions when the object is realized through sense-making actions in an activity. The process by which the object is set in motion, the process that changes the object from its potentiality to its actuality through sense-making actions, can be described as ascending from the abstract to the concrete (Davydov, 2008; Radford, 2015).

Algebraic reasoning

In the algebraic tradition, algebraic reasoning as an aspect of algebraic thinking, is explained as the student's ability to: 1) understand patterns, relations, and functions; 2) represent and analyse mathematical situations and structures using algebraic symbols; 3) use mathematical models to represent and understand quantitative relationships; and 4) analyse changes in various contexts (Cai & Knuth, 2011). More specific explanations of algebraic thinking are offered by, scholars such as, Davydov (2008, p. 148) and Radford (2015; 2018). Davydov is focusing on relations in, and between, quantities. He suggests that algebraic thinking entails: 1) an introduction to relations between quantities; 2) discovery of the ratio relation in quantities; 3) the formation of all real numbers; and 4) discovery that any mathematical operation has a structure. Radford suggests that algebraic thinking entails: 1) reasoning about indeterminacy; 2) denotation of this indeterminacy using natural language, gestures, unconventional signs, or a mixture of these; and 3) analyticity in which indeterminate quantities are treated as if they were known numbers. These explanations are broader than the traditional explanation of algebra as simply constituting general arithmetic. Using Davydov's and Radford's explanations, algebraic thinking can be seen as opportunities for students to analytically deal with unknown quantities and unknown numbers in relation to different objects of knowledge.

For these analyses, undertaken by even the youngest students, Radford (2014) suggests a classification of three forms of algebraic thinking: non-symbolic, contextual, and symbolic. In *non-symbolic algebraic thinking*, students analyse, for example, an unknown quantity, using words from everyday life and argue for ways to implicitly solve a mathematical problem, using, for example, gestures and rhythms (Radford, 2014). Here, students can reason analytically and operate with unknowns without using numerical symbols, with the help of gestures and words they usually use. Students' *contextual algebraic thinking* comprises symbols and rhythms in relation to an explicit subject content. Finally, students' *symbolic algebra thinking* involves alphanumeric formulas. All three types of algebraic thinking are regarded as vivid narratives instead of just calculations using formulas (Radford, 2014); they can be used as descriptors to identify the sense-making actions students do during their analytical work of unknown quantities and the relationships in and between these quantities.

Fractions

Difficulties and misconceptions in relation to developing number sense in regard to fractions have been presented in many previous studies (e.g., Davydov & Tsvetkovich, 1991; Nunes et al., 2009). Davydov (2008) argues that number senses regarding whole numbers and rational numbers should develop out of the same context and cultural–historical tradition of numbers. Thus, comparing quantities is rooted in human cultural–historical work, and accordingly, comparing and measuring are important activities in developing a number sense (Davydov, 2008). To enhance number sense regarding fractions that arises out of measuring, Davydov and Tsvetkovich describe three key concepts to be addressed: a quantity to measure (e.g., quantity as a length, weight, volume, or area), a unit of measure, and a smaller unit with which to measure the unit of measure. For whole numbers, the quantity to measure can be described using an integer, while for rational numbers the quantity must be described using an integer and a remainder, with the remainder being described in terms of the relation between the numerator and denominator.

DATA AND METHODS

The data used here consist of transcripts from one lesson in a research project to develop students' understanding of fractions (Eriksson, 2015). At the core of the project was a collective iterative process (involving me as a researcher and one of the three participating teachers) of designing and redesigning a learning activity. The twenty participating students were 9, 10, and 11 years old, more than half of them were newly arrived in Sweden. The lesson drew on the so-called El′konin–Davydovs' mathematical curriculum with learning activity as a framework (Davydov, 2008). The actual learning tasks in the lesson were inspired by Davydov and Tsvetkovich (1991).

To develop a learning activity, it is important to start with a common problem that the students and the teacher can collaboratively identify. In a "pre-lesson" before the research lesson, the students first used the measurement activity to represent equalities using Cuisenaire rods. These equalities were presented in the form $A = WB$, where rod

A could be described by a whole number (*W*) of rods (*B*). When the students were asked to design their own equalities, in which the length of one rod equalled the length of a whole number of just one other kind of rod, the students faced the problem that the length of rod *A* could not equal any whole number of any other kind of rod. One of the measures constructed by the students in this pre-lesson was to compare a black rod with red rods (Fig. 1), while another was to compare a black rod with yellow rods (Fig. 2).

Fig. 1: Black = 3 red + ½ red

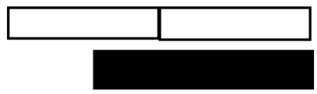

Fig. 2: Black = 1 yellow + 2/5 yellow

To find the quantity of the red and the yellow rods, the problem was to identify a fractional part of these rods. To solve this problem, the students had to identify the length to be measured (i.e., a black rod), the unit of measure (i.e., the red or yellow rod), and the quantity of this unit. The result had to be presented in the form $A = \mathrm{W}B + rem$, where *A* is the length to be measured, *B* is the unit of measure, *rem* is the remainder when constructing the equality, and *W* is the number of whole units of measure (compare Davydov & Tsvetkovich, 1991).

A first step in the analysis was to describe the activity as a narrative description suggested by Eriksson (2017). This description was done in relation to fractions while the students and the teacher were measuring a black rod in terms of red or yellow rods (see Figs. 1 and 2). A second step in the analysis was to identify the potentiality of the object of knowledge and its actualities, as suggested by Radford (2015; 2018). Four central sequences of the transit from potentiality to actuality were identified in the narrative description, coded 1, 2, 3, and 4. Analytical questions asked in this step were: What changes of the object of knowledge can be identified? and, What sense making actions made these changes possible? The last step of the analysis focused the sense making actions in these sequences. Analytical questions asked in regard to the sense-making actions were: What actions can be identified as an emerging algebraic reasoning described by Radford (2015)? and, How, was the unknown fractional part elaborated?

ANALYSES

This section presents the analysis of the four identified central sequences, which are illustrated using excerpts and observations from the lesson.

Sequence 1:

When the students entered the classroom, one of the measurements from the pre-lesson was attached to the whiteboard (Fig. 1). As the teacher drew a number line on the whiteboard; the students reacted by telling the teacher that they remembered the problem with that measurement.

Teacher: Then, what was the problem? Do you remember what your problem was?

[After a short whole-class discussion]

Bayar: It is three and a half. There is three of these, and then there was one rod that was longer, so if you add one more it would be longer than the black rod. But if you just add a little bit, they would be the same length. If you add one half.

[While saying this, Bayar was sitting at his desk, pointing at the depicted measurement on the whiteboard. First he moved his hand in three distinct gestures, pointing at the three whole units (one, two, three). Then he moved his hand up and down in small movements, and at the end of his comment, he put his other hand in the middle of the up and down movement.]

In this sequence, the students changed the object of knowledge from being just a measurement problem involving two lengths that represented inequality, to three whole units and one unit "that was longer". The sense-making constituting this change was conveyed by Bayar, who first pointed at the three whole units (one, two, three), saying: "There are three of these". Then, he indicated that something was different about the fourth unit (moving his hand up and down, and putting his other hand in the middle of the gesture), saying: "If you just add a little bit, they would be the same length". In this sequence, Bayar and the other students also indicated that there was an additive relationship between the whole and fractional part ("if you just *add* a little bit"). Altogether, these actions can be identified as illustrating emerging algebraic reasoning in relation to the students' analysis of the unknown quantity in the measurement problem. A reasoning expressed using non-symbolic means, through natural language and gestures without using numerical symbols to refer to the fractional part. The students were initiating theoretical work about the abstract additive relationship involving a mixed number ("add just a little bit"). The teacher used this as a resource to further develop the students' theoretical work about fractions.

Sequence 2:

[The whole-class discussion continues.]

Teacher: How can you know that? How can you know that you have to add a half?

Leart: Maybe, we can measure? We do need to measure the red rod, don't we? And the little bit left?

In this second sequence, the object of knowledge was changed, or developed, from the suggestion "add a little bit" to the further analysis of the fractional part, i.e. that they had to measure the unit of measure in order to understand the fractional part. The emerging algebraic reasoning was identified as a further step in analysing the unknown fractional part, in which measuring even the unit of measure was suggested.

Sequence 3:

Teacher: We have to return to our problem. What are we going to do?

Dana: We are going to tell the black …

Chaid: The black is the whole one and there, it is a little bit more.

Evin: Last lesson, this was our problem …

Ami: We need something that is smaller than the red ones to measure with.

[When Ami suggests this, the teacher takes two white rods, a shorter unit than the red rod, and puts them on the board beside the last red unit.]

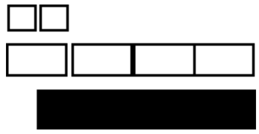

Fig. 3: The shorter (white) unit was put in the measurement on the white board.

In this third sequence, the object of knowledge was changed, and further developed, from a need to measure, to a need for a unit shorter than the original unit of measure. The actions that made sense of that change were Ami's suggestion to use a smaller unit and the teacher's modification of the measure (Fig. 3). The emerging algebraic reasoning can thus be identified as the analysis of the unknown fraction part of the original measurement unit with the new shorter unit.

Sequence 4:

[The teacher initiated a discussion of how to represent this number, starting with the students' comment "a little bit more", which was used as an emerging model of the fractional part of the number. The teacher wrote: "*Black* = W + *a little bit more*" on the board. A further step of the learning model, comprised of the idea to use a shorter unit to measure the unit of measure, was written by a student: *B* = W + *white rod/white rods*. A third learning model was then suggested and written by the teacher: *B* = W + m/n. This model was rejected by the students, who instead suggested: *B* = W + p/a. According to the students, *W* is the whole unit, *p* is the part of the last small unit needed to achieve equality, and *a* is all the small units used to measure the original unit of measure. Next, measuring the black rod with yellow rods (see Fig. 2), the students developed the learning model: *B* = W*y* + (p/a)*y*. The final learning model was realized to be even more connected to the empirical measure. The students suggested: *Black* = W*yellow* + (p/a)*yellow*.]

Through these four sequences, the object of knowledge underwent a stepwise emergence of a learning model containing algebraic symbols. In the first step (*Black* = W + *a little bit more*), the actuality object of knowledge initiated by the students was the additive relationship between the integer "W" and the fractional part of the number implied by "*add a little bit more*". In the second step (*B* = W + *white rod/white rods*), the relationship between the numerator and the denominator emerged. The actuality object of knowledge in this step was the multiplicative relationship between the numerator and denominator in the fractional part of the number. In the third step of the model (*B* = W + m/n), the symbols "m" and "n" were suggested as they are common algebraic symbols for the numerator and denominator (according to the teacher group that planned this lesson). However, in the fourth step (*B* = W + p/a), the students instead wanted to use algebraic symbols with a contextual meaning, i.e., "p" for the partial unit they needed to develop equality and "a" for all the small pieces into which the unit of measure had to be divided into. Here, the students used symbols in an algebraic way, to represent relationship between variables in a general way. By doing so, they connected the symbols to the context of the fractional part of the number, "p" (i.e., numerator), and "a" (i.e., denominator). These symbols emerged from the multiplicative relationship in the fractional part of the number. As suggested by the students, the learning model then became increasingly concrete, i.e., increasingly connected to the specific measures addressed, as they were embodied by the rods. Furthermore, in measuring a black rod using yellow rods, the students first suggested the learning model B = Wy + (p/a)y and then *Black* = W*yellow* + (p/a)*yellow*. The students' algebraic reasoning can thus be identified as the analysis of the additive relation between the integer and the unknown fractional part of the number, and as the multiplicative relationship in the unknown fractional part.

RESULTS AND CONCLUDING REMARKS

In this paper, algebraic reasoning about fractions was identified through sense-making actions when the students and the teacher analysed the unknown fractional part of mixed numbers (Davydov, 2008; Radford, 2014). These sense-making actions focused on different relationships in a mixed number, such as an additive relationship between the integer and the fractional part, and a multiplicative relationship in the fractional part of the number (Davydov, 2008). The fractional part of the number was first analysed using natural language (e.g., "add a little bit") and gestures when the students and the teacher were pointing at the various Cuisenaire rods. Such sense-making, according to Radford (2015), can be described as non-symbolic algebraic thinking. I noticed how the students identified the additive relationship between the integer and the fractional part of the number (Davydov, 2008; Davydov & Tsvetkovich, 1991), and how, through further analysis of the unknown fractional part, the students suggested that a smaller unit should be used to measure the unit of measure (Davydov & Tsvetkovich, 1991). This suggestion, to measure the unit of measure, made sense to the students as they were exploring the multiplicative relationship inherent in the fractional part of the number, which was further developed when a student wrote "one rod/two rods" on the board. The learning model for the unknown fractional part, containing algebraic symbols emerged from, $B = W +$ *a little bit more*, leading to the more concrete model connected to a specific measure, *Black* = *W*yellow + (*p*/*a*)yellow. Following Davydov (2008) and Leontiev (1978), we can describe the students' elaborations of the learning model as processes of objectification; as ascending from the abstract to the concrete. The students' suggestions of what algebraic symbols to use in contextualizing their algebraic reasoning manifest Radford's (2015; 2018) descriptions of contextual algebraic thinking. In this lesson, the object of knowledge—the unknown fractional part of the number—emerged through a process of realizing the potentiality of the object (i.e., add a little bit more), through sense-making actions (i.e., algebraic reasoning in a contextual manner), to various actualities of the object.

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MULTIPLICATIVE STRUCTURES IN ELEMENTARY SCHOOL MATHEMATICS: RELATIONAL APPROACH

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In this theoretical essay, we critically analyze some multiplicative structures identified by researchers and practitioners in the field of teaching elementary school mathematics (grades 3-5). Drawing upon Davydov's theory of developmental instruction, we use the relational perspective and propose another classification and graphical representations of multiplicative structures. We suggest that the new approach may better support student understanding of multiplicative relationships and at the same time contribute to the foundation of their algebraic thinking.

INTRODUCTION

The teaching and learning of problem solving, a fundamental pillar of school education, always attracts researchers' close attention. Problem-solving activities formally debut at the beginning of schooling where students deal with simple word problems requiring one arithmetic operation for their solution. In order to support effective teaching and learning of problem solving at this stage, researchers analyzed simple word problems and their semantic structures and produced typologies of additive word problems (requiring one addition or subtraction operation) and multiplicative word problems (requiring one multiplication or division operation). However, theoretical clarification about the mathematics behind simple word problems did not lead to the elimination of teaching and learning difficulties. What's more, recent works about early algebra and modelling bring new perspective in the area of simple word problem solving.

In 2011-2014, responding to the practical needs of teachers in our region (Quebec), we conducted a research project focusing on additive problems and their teaching in grades 1 and 2. We aimed at facilitating students' learning to solve such problems as well as at developing their mathematical thinking. After having critically analyzed the literature concerning additive word problems, we proposed the existence of two distinct paradigms of research in the area: the operational paradigm and the relational paradigm. The operational paradigm recognizes arithmetic operations as the basis for understanding real world situations (or word problems) involving adding, removing, comparing, equalizing, sharing etc. The relational paradigm, however, preconizes the holistic and flexible understanding of simple relationships between three quantities as the foundation for solving such problems. Drawing upon Davydov's (1982) definition of the additive relationship as "the law of composition by which the relation between

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two elements determines a unique third element as a function" (p. 229), we tried to root the teaching and learning of additive problem solving in students' holistic and flexible understanding of this relationship. We also use a specific graphical representation of the additive relationship to allow students' modelling additive problems based on their understanding of the relationship involved. This new epistemological perspective proved to be fruitful allowing for the design of quite successful teaching strategies. The teachers who participated in the study reported that they could not imagine returning to the old way of teaching (based on operations and key words) (Savard et al., 2018).

Since 2016, an extension to our study has been aimed at examining the multiplicative word problem solving in grades 3-6. To date, following an in-depth analysis of available typologies of multiplicative situations (or word problems), their sematic structures, and their graphical representations, we analyzed the multiplicative situations from the relational perspective.

In this paper, we discuss simple multiplicative word problems, their structures and typologies, and their graphical representations as these are treated in the literature. We then discuss, in more detail, the relational perspective. We conclude by suggesting that specific representations of multiplicative structures, coherent with the relational paradigm, can enhance students' thinking mathematically and their learning to model and solve problems.

MULTIPLICATIVE STRUCTURES AND THEIR REPRESENTATIONS

Discussing understanding and representations in mathematical problem solving Vergnaud (1983) distinguishes two possible directions in their analysis: implicit representations the solver can have of the problem and explicit representations the solver might create to communicate the important elements and retained operations in form of graphic drawings and letter expressions. (Vergnaud, 1983, p. 33). Vergnaud suggested that the former informs the latter. Thus, researchers can observe and study students' graphical representations of problems in order to understand their implicit internal representations.

Some researchers (e.g. Rockwell, 2012; Mancl, 2011) also propose to explicitly teach schemes and graphical representations to students, especially to those having difficulties in mathematics, and claim that such practices facilitate problem solving. Some teachers' manuals (e.g. MEO, 2008; MELS, 2009) and official documents that guide teaching practices provide various typologies of multiplicative structures together with their graphical representations claiming that these representations should not only support teachers' understanding of these structures but also guide their teaching.

As part of our extended research, we examined these sources and analyzed the multiplicative structures and the representations provided in them in order to identify the kind of thinking they might potentially evoke in learners if used explicitly within the teaching-learning activity.

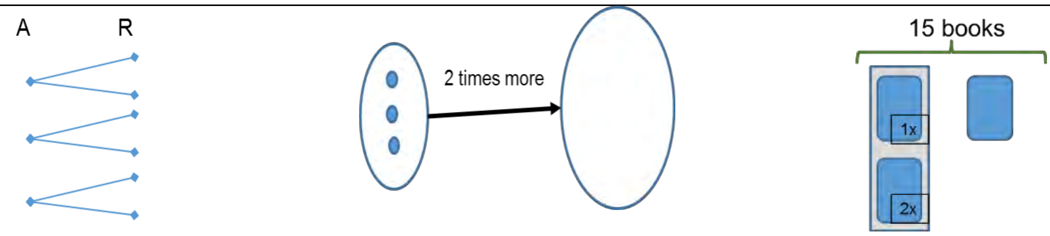
Some of the problem types (or categories) studied reflect mainly mathematical structures (e.g. Cartesian product, mapping rule, equal groups) while others pay special attention to actions (e.g., rate, multiplicative change, sharing), disposition (rectangular disposition), comparison expression (e.g., times more, times less) or what is unknown or invariant.

In addition, representations of multiplicative structures are also diverse and reflect different aspects of the represented situation. For example, Nunes, T., & Csapó, B. (2011) mention the following problem:

> Together Rob and Ann have 15 books (quantity). Rob has twice the number of books that Ann has (or Ann has half the number of books that Rob has) (relation). How many books does each one have? (p. 29)

This problem presents a multiplicative structure belonging to the multiplicative comparison category. The authors propose to represent this situation as shown in Figure 1 (a).

Adapted from Nunes, T & Csapó, B, (2011).	Adapted from MELS (2009).	Adapted from Van de Walle & Lovin, (2008)



(a)	(b)	(c)

Figure 1: Representations of the book problem

A close look at this representation makes it clear that each point (extremities of arrows) in column A represents one of the books Ann has, and that each point in column R represents one of the books Rob has. The number each has, being unknown, makes it impossible to fully represent the situation.

Other sources propose different representations for the multiplicative comparison (b). In this representation, each element of the known set is represented by a dot and the comparison expression is shown as is in the problem (2 times more).

Using representations (a) and (b) to solve problems may become cumbersome if large numbers are involved or if the multiplicative structure is a part of a more complex structure, as in the book word problem above.

The representation (c) above, however, carries different characteristics. It shows repetitions of a set equal to the referred set and uses number notation to specify the comparison relationship. It does not show elements of singular sets. As such, it has an advantage over the first two representations of multiplicative structures because it can represent sets of any magnitude and can be combined with other representations to deal with more complex problems such as the book word problem described earlier.

Some other representations we found (e.g. MEO, 2008; Vergnaud,1983) *encode* the situation as an operation, using mathematical symbols or arbitrary icons. They do not represent objects, sets or relationships in *analog way* thus making it difficult for the observer to create a link of analogy between the problem and its representation.

The neuroscientist Yan Robertson (2017) explains that solver might ignore the essence of a problem by paying more attention to its superficial aspects. He explains that this behavior is quite natural at the beginning of the learning process, he also suggests the important role analogy and mental schema play in analyzing and solving a problem. He writes:

> It is obvious from the preceding discussion that analogies can play an important role in influencing thinking. One way that analogies can operate is by activating a schema that in turn influences the way we think about a current situation. (p. 83)

The same, we argue, is true for representations: individuals represent the things they pay attention to and the way they understand it, using available knowledge and mental schemes. Thus, if the thinking about objects or about exact number of objects is what students pay attention to, they will probably try to use schemes a) or b). Once chosen by a solver, the type of representation shapes the entire thinking process. Trying to represent each object (e.g., each individual book), students will be blocked and unable to solve the problem. Representation (c), however, attracts solvers' attention directly to the relationship between sets; it is more flexible and can help to understand the entire problem at hand in its complexity.

We can conclude that some representations might lead to dead ends, thus becoming counterproductive in terms of learning because they do not support cases that are more complex. In some conditions, they can act as didactical obstacles preventing students from constructing of new representation of a problem. We propose that the explicit use of such limited and limiting representations by the teacher may evoke spurious elements in students' minds and eventually derail their attention from the multiplicative structure we would like them to recognize and be able to use to solve the problem.

In the context of Quebec, as well as in some other Provinces and countries around the world, it is generally accepted that students should develop their own representations to solve problems. However, speaking about "ratio" and "rate" Thompson (1994) argues:

> [H]ow one might classify a situation depends upon the operations by which one comprehends it. In Thompson (1989) I illustrate how an "objective" situation can be conceived in fundamentally different ways depending on quantitative operations available to and used by the person conceiving it (p. 16).

Indeed, according to Robertson (2017), many beginners turn their attention to irrelevant elements of the problem at hand, and therefore, the representations they may construct can lead them to dead ends. This means that allowing students to construct their own representations is not as effective a teaching strategy as many believe because students may fail to construct representations that reflect the mathematical relationship in the problem at hand. Based on Thompson's (1994) ideas, we suggest that for a student to construct a representation of a multiplicative structure (for example multiplicative comparison) he or she needs to possess the concept of this particular relationship to which we turn next.

RELATIONAL PERSPECTIVE

In his seminal work, Davydov (1982) put forward the idea of quantitative relationships as mathematical concepts that we need to teach and learn in elementary school, even prior to numbers. He argues that the concept of number appears from the multiplicative comparison of two magnitudes (or quantities), one playing the role of unit of measurement and the other being measured. In their experiments, researchers introduced students first with situations of comparing and measuring water, surfaces, and ropes through the manipulation of real objects rather than stories or word problems. They also used specific graphical and symbolic representations to analyze and discuss these situations with students. They then used the developed representations to support word-problem solving.

Based on Davydov's ideas, researchers in other countries successfully implemented the measuring approach in developing number concept with elementary students (e.g. Dougherty & Slovin, 2004). This growing research suggests that the relational approach might enhance students' mathematical thinking and particularly their understanding of unit (Barrett et al., 2011). As far as we know, the multiplicative relationships and their graphical representations did not yet attract adequate attention in the context of simple word problems. While graphical representations can be found in many textbooks and teacher guides, it is not yet clear how exactly teachers use these representations in classroom. Thus, the role these representations might play in the learning process is not theoretically developed.

According to the relational perspective, understanding a word problem means that students recognize the quantitative relationships involved. Sandra P. Marshall (1995) argues that it is possible to find the *basis set of schemas* (analogous to basis set of vectors in a space) to be able to understand and describe all and any multiplicative problem in a domain. What, then, can be a minimal set of multiplicative relationships allowing for the understanding of all multiplicative problems of the elementary school?

SIMPLE MULTIPLICATIVE RELATIONSHIPS AND THEIR REPRESENTATIONS

Studying works of Davydov, Thompson, and Vergnaud yields the identification of the following relationships.

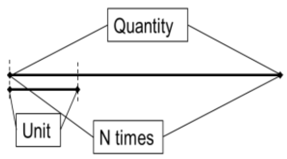
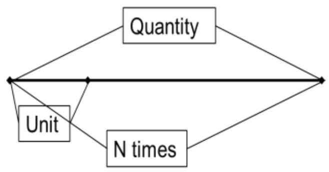
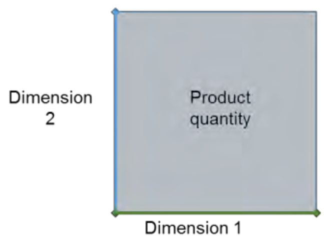
Relationship	Description	Representation	Examples
Multiplicative comparison or measurement	This relationship can be used if one quantity can be measured (or compared in multiplicative way) against another quantity that is physically distinct from the first one yielding a number whether it is known or unknown.		Max has three times as many marbles as Maya. Max' shoe measures twice the Maya's shoe. How much is Maya younger than Max?
Multiplicative composition	This relationship can be used if one quantity is composed of a number of equal parts (number can be rational as well).		Max has many boxes with the same number of marbles in each. A car moving with a constant speed made a certain distance in a certain time.
Cartesian product	This relationship can be used if all three elements of the multiplicative relationship have different physical origins and none of them can be seen as a pure number or as a unit of measurement.		One uses a number of skirts and a number of blouses to create costumes. One evaluates a rectangular area in relation to its length and width.

Table 2: Basic multiplicative relationships and their graphical representations.

Many pedagogical sources describe a proportion among multiplicative structures. Obviously, the proportional relationship deserves special attention. This relationship can be used when one quantity being measured by the second yields the same number as the third being measured by the fourth. The following example presents such a relationship.

> A glass of cocktail contains 40% of apple juice. For any two particular amounts of cocktail, the juice part measures the same (each time, we consider the amount of cocktail as a unit of measurement).

However, proportional relationship, in our view, is not a simple relationship so we do not discuss it here. This set of relationships is not like orthogonal basis vectors, because one can employ two or three of them (one at a time) to interpret the same simple word problem. However, this set is sufficient to interpret all multiplicative problems from the repertoire of elementary school. All graphical representations we propose show the underlying multiplicative relationship—thus highlighting the essential mathematical idea of the problem. At the same time, they do not show irrelevant elements such as objects or their numbers (these elements are irrelevant when we are looking for the arithmetic operation to figure out the unknown element of the relationship; they however, can be relevant when carrying out the chosen mathematical operation). The explicit use of these representations by the teacher may help not derail students' attention from the essence of the problem.

Representing a quantity by a segment or a lengthy rectangle has many advantages. First, there is a way to imagine objects arranged into a line. This mental organization allows for the representation of any number of objects as well as the preservation of the meaningful link between the initial situation and its representation. Second, it helps to represent an unknown quantity because it can be imagined as a line of objects. Finally, all elements of the relationship can be visually represented and simultaneously analyzed, which, in turn, helps to derive the arithmetic operation to find out the unknown element.

The representations we propose can be easily combined to describe more complex situations. As we mentioned above, many textbooks in some countries use visual representations to support students' problem solving. Yet, why and how they might help is not theoretically clear. We support Davydov's idea that the concept of multiplicative relationship(s) is to be taught and developed by students prior to solving complex word problems. Therefore, we believe that it is more effective to conceptualize the activity of modelling, discussing, and solving simple word problems through development of the concept of multiplicative relationship. This activity is not *problem solving* per se, because students should not just apply their knowledge of multiplication and division to find a solution to a problem. Rather, this activity is learning about basic relationships and developing mental schemes that will eventually support analysis, modelling, and solving of situations that are more complex, thus allowing for a real problem solving.

We would like to share our classroom experience in developing multiplicative relationship with students. Space limitation precludes a description of such work but we are looking forward to presenting these at the conference.

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NATIONAL PRESENTATION

MATHEMATICS EDUCATION RESEARCH IN SWEDEN: NATIONAL PRESENTATION AT PME 42

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INTRODUCTION

If Umeå University had waited one more year to arrange the annual meeting of the international group for the psychology of mathematics education, it would have coincided with the 100-year anniversary of what is usually considered the first Swedish PhD in mathematics education research. K. G. Jonsson received his PhD from Uppsala University in 1919 on a dissertation treating arithmetic problem solving based on interviews and observations of three pupils. Despite the early start, mathematics education research did not develop into a research area in its own right in Sweden until the last decade of the twentieth century, even later than many other countries, including our Nordic neighbors. In fact, the history of mathematics education research in Sweden, which is tightly connected to school-related development work, is rather peculiar. Schoenfeld describes the international development:

> Until the latter part of the 20th century, research in mathematics education was an orphan discipline, with neither an identity nor a home. It had roots in philosophy, although those were often not explicitly recognized; it was shaped by mathematicians and psychologists; it drew on empirical methods from the fields of statistics, measurement, and psychology. Historically, many education researchers focused on mathematics education, but those practitioners had no center of gravity—no professional organizations, no journals of their own. Then, just as education as a field had become professionalized in the early 20th century, mathematics education came together as a discipline in the 1960s and 1970s. (Schoenfeld, 2016, p. 505)

We will tell our story of mathematics education research in Sweden by looking at the present situation as well as the historic development leading up to where we are today.

At the outset, it must be said that it is difficult to review the historical development and current state of a research subject in one country without forgetting or missing important perspectives, events, phenomena, or persons, which we have probably done. In preparation for writing the present text, we read previous descriptions of Swedish research in mathematics education, most notably Björkqvist (2003), Strässer (2005), Sriraman et al. (2010), and a review of Nordic research compiled by Rönning (In preparation), in particular, a longer earlier manuscript of Rönning's text. We emailed or talked to people with long histories in the field: Gerd Brandell, Christer Bergsten, Göran Emanuelsson, Arne Engström, Barbro Grevholm, Bengt Johansson, Johan Lithner, and Astrid Petterson. We also contacted at least one person at each university

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 273-297). Umeå, Sweden: PME.

with activities in the mathematics education area and asked for a short description of the local research. Finally, using only the titles, all the Swedish PhDs in the area were classified according to content.

Before moving on to describe the research, we give a short description of the development of the Swedish school system.

THE CURRENT EDUCATION SYSTEM IN SWEDEN AND ITS HISTORY

The core of the Swedish education system is the compulsory school (*grundskola*), which children enter when they turn seven years of age. The comprehensive school begins at nine years and is a non-tracking system. It has basically looked the same since 1972, when the ninth year became compulsory, though curriculum reforms have been common, and the governing of school has undergone some radical changes that we will not deal with here. The roots of the system go back to ideas about a comprehensive school system, common for all classes in society, formulated in the late 19th century by politician Fridtjuv Berg. Although some steps toward such a system were taken in the early 1900s, the most important transitional stage came around 1950, when experiments with a nine-year comprehensive school were initiated in some municipalities. It was, in fact, already decided by the parliament that such a system should be introduced for all children throughout Sweden, but experimentation with the form as well as expansion to all of Sweden took until 1972.

The year before compulsory school, intended for children between six and seven years of age, has had a particular place in the school system since 1976, when all six-year-olds were given the right to a free 15-hour-per-week general preschool. In 1998, the name *preschool class* was introduced, and the age group was incorporated into schools rather than preschool and child-care organizations. Beginning in the autumn of 2018, preschool class will be compulsory for all children, which in practice means that Sweden will move from a compulsory nine-year school system to a 10-year system.

In addition to formal schooling, Sweden has a well-developed preschool system offered to children from one year of age until they enter preschool class. Preschool in Sweden has a long history. Through private donations, precursors of the current preschool began in Stockholm in 1854 with the aim of taking care of poor people's children, enabling both parents to work. From 1943, child-care institutions became partly subsidized by the state. A governmentally assigned committee, *Barnstugeutredningen*, worked from 1968 to 1972. Coining the term *preschool* (*förskola*), they suggested that the pedagogical responsibility of preschool should be strengthened. In 1998, preschool was regulated through a national curriculum, where certain goals for mathematics were prescribed, and these goals were further strengthened in 2010. Historically, the Swedish preschool has had a strong social pedagogy tradition, as opposed to a pre-primary tradition (Bennett, 2005), but seems presently to be adopting a dual approach that mixes pre-primary and social pedagogy (Sheridan, Williams, Sandberg, & Vuorinen, 2011) with a particular focus on play (Synodi, 2010).

At the other end of compulsory school, Sweden has an upper secondary school system called *gymnasieskolan*. Upper secondary education was first formally regulated in Sweden in 1856, when two programs were introduced, *latinlinjen* (humanities and social sciences) and *reallinjen* (mathematics and natural sciences). In 1971, this system was replaced with the current one, which has been restructured two times since then (Skolverket, 1994; 2011). From 1862 to 1969, upper secondary education ended with a formal test, the student exam, which served as the main way to assess qualification for university studies. After 1969, the student exam was replaced with a grading system in which students were awarded a final grade in each subject on a 1–5 scale following a Gaussian distribution. To manage this system, a set of national standardizing tests was used. Hence, the grading system was norm referenced. This system was abolished in 1994, with the introduction of a completely new curriculum and a criterion-based grading system. This system has since been reformed once, but the basic ideas remain. In its current form, the core of the Swedish upper secondary schooling is 18 national programs, all three years long. Each program is built on courses with a hierarchical structure. Mathematics courses are arranged in three strands: 1a and 2a, adapted to vocational programs; 1b-3b, adapted to economy, humanistic, social science, and esthetic programs; 1c–3c, adapted to science and technology programs; and, in addition, courses 4 and 5 and a specialization course. Different programs include from one to five of these courses as obligatory (Skolverket, 2017).

RESEARCH AREAS IN FOCUS

Previous overviews of Swedish research (Bergsten, 2002, 2010; Björkqvist, 2003, Rönning, in preparation; Strässer, 2005) have argued that is it hard to pinpoint some particular Swedish flavor of research or even (with some exceptions) characterize areas that are particularly active in Sweden. These reviews come up with different lists not only because the target is moving but also because there are many different relevant categories and ways to classify particular projects that often deal with more than one issue. First, we present an overview of PhD dissertations that were classified by Kilhamn according to their titles using some rather general categories. We then go on to discuss some patterns that were identified in the reports from different university environments.

Analysis of dissertations

The graph in Figure 1 is based on an analysis of 147 dissertations categorized solely according to their titles. The dissertations were initially categorized using five a priori categories: 1) communication and language, 2) reasoning, 3) assessment, 4) ICT, and 5) mathematical content. The categories are hierarchical so that each dissertation was placed in the highest possible category. In total, 85 dissertations fitted the a priori categories, with only six possible overlaps. The remaining 62 dissertations were then further analyzed and placed into categories emerging from the data. Among these, we found a focus on mathematic teaching and students in general, student identity, organization of the mathematics classroom and curriculum, and a few others that may include future topics not yet identified.

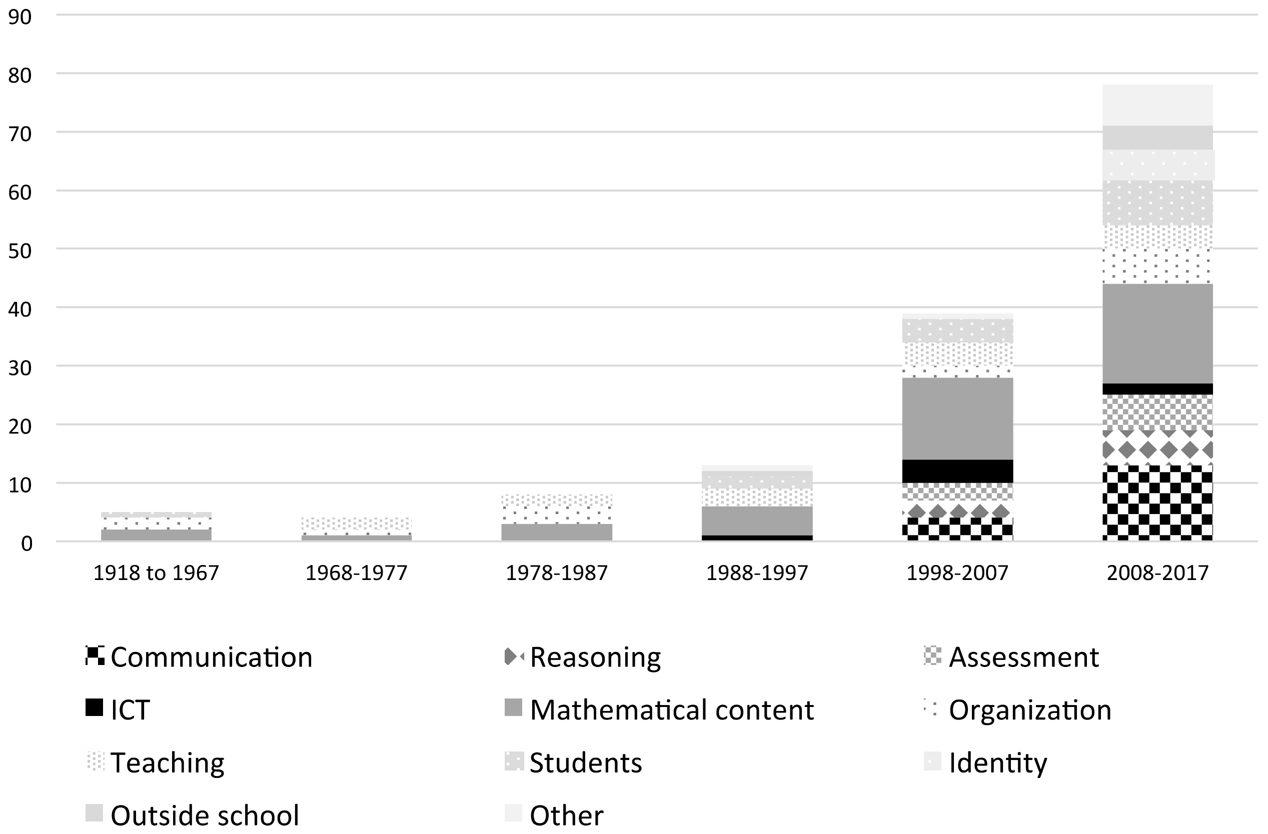

Figure 1: Swedish mathematics education PhD dissertations 1918-2017.

One conclusion that can be drawn from Figure 1 is that the number of dissertations per year has grown almost exponentially since the end of the nineties. Bergsten (2010) described the increase from about 1990 to 2009 as an "explosion," and we can conclude that the development has been even stronger since. Another impression from Figure 1 is that the focus of the research seems to vary a lot with dissertations spread over a large number of areas. However, two areas seem to have drawn more attention than others: mathematical content and communication.

Mathematical content ended up being the most common category, despite being slightly disadvantaged by the classification method. This seems to contradict Björkqvist (2003), who in his analysis claimed that *stoffdidaktik* (subject matter didactics) is a missing perspective in Sweden. However, Strässer (2005) remarked that among the 22 dissertations he classified as dealing with specific mathematical content, all except one used this mathematical focus to study some other didactical phenomenon. Although we have not analyzed all of the content-oriented dissertations produced since Strässer made his list in 2004, we can make similar reflections based on a few samples. Some, like Hansson (2006), have a clear focus on student teachers' views of the mathematical content. Some have a crossover character, like Nilsson (2006), where neither general theories about conceptualization nor the particularities of the mathematical content—probability—can be exchanged. Others, like Sjögren (2011), included some mathematical content, but the bulk of the arguments concerned something else, in Sjögrens case philosophy. However, it does seem as if, in general, and in line with

Brandell's (2010) conclusion, there has been a slight increase in dissertations that deal with mathematical content. One reason for this might be the particular conditions of the national graduate school that ran from 2001 to 2006. We describe these conditions in our historical account.

Communication and reasoning is also focused on more than other areas in Swedish dissertations in mathematics education. Since research addressing reasoning tends to be closely related to research addressing communication, we discuss the two areas jointly.

The wide range of perspectives within these categories includes issues of power dealt with by invoking Foucauldian perspectives (Skog, 2014), didactical analyses of teacher-student dialogues (Löwing, 2004), theory-driven analyses of student collaboration (Nilsson, 2006; Ryve, 2006), and the creation and use of a classification framework for different types of mathematical reasoning (Bergqvist, 2006; Liljekvist, 2014; Lithner, 2001; Norqvist, 2016). It also includes linguistic perspectives in textbook research (Segerby, 2017; Österholm, 2006). The growth of research on communication and reasoning cannot simply be derived from the creation of the national graduate school, so why is it happening? One hypothesis is that research was influenced by the national curriculum from 1994, in which a competence type goal structure was introduced, inspired by the process goals in National Council for Teachers of Mathematics standards (NCTM, 1989; 1991). This perspective was strengthened in the 2011 curriculum revision, now also influenced by the Danish mathematical competencies project, Kompetencer og Matematiklæring (Niss, 2004) which includes a focus on communication and reasoning. It is possible that this also sparked research inte-rest in such issues. Another hypothesis is that it simply is an effect of more general interna-tional trends, that is, the social turn (Lerman, 2000).

Assessment is another category of research that has grown very strong in recent years. As we describe later, if we go back several decades, Sweden has had a very strong tradition regarding tests and testing. There are two environments (in Umeå University and Stockholm University) that work with the construction of national tests in mathematics. Classical issues of testing are, however, mostly absent from our list of dissertations, with the exceptions of Nyström (2004) and Eklöf (2006), both produced in the Umeå University research group around pedagogical measurement. Common perspectives are, instead, formative assessment, or studies of the effects of tests on pupils, teachers, or school systems. Perhaps the interest for formative assessment (e.g., Black & Wiliam, 1998) is not surprising, given the international influence of this perspective, particularly when it comes to professional development.

Reports from institutions about current research

Some of the trends identified above return when we look at how institutions descrybed their research to us. For example, different perspectives on language, communication, and reasoning were mentioned by Umeå University, Mid Sweden University, Malmö University, and Örebro University, among others. The perspectives include using sociolinguistics (Malmö University) or other linguistic techniques (Umeå University) to study textbooks and to relate linguistic forms of arguments to how students understand them. In Örebro University, theories from the philosophy of language (inferentialism) are used to analyze student collaboration. An Örebro-Uppsala collaboration also works with the language of textbooks. The strong focus on communication and reasoning is in line with what we saw in our analysis of dissertations.

The textbook work mentioned above is part of a more general textbook category among the reported research projects. Textbook analyses have been oriented toward specific content areas, like proportional reasoning (Ahl, 2016), or in the form of comparative studies of textbook structures and content (Hemmi et al., 2013). Mälardalen University also houses a very ambitious development project aiming to produce a research-based textbook series. Textbook research is, as mentioned, also carried out at Umeå University, Malmö University, Örebro University, and Uppsala University. Given that Swedish teachers are known to rely heavily on textbooks, it is perhaps not surprising that many researchers focus on this, both developmentally and analytically. TIMSS data suggests that 97% of Swedish students in grade 8 have mathematics teachers who plan their instruction by following the textbook (Mullis et al., 2012). Also, some research that initially seems to be subject-matter oriented, like the project on algebra in Uppsala University, is largely based on analyses of textbooks and other curriculum materials and national curriculum documents. Some of the research on textbooks in Sweden is reviewed in Jablonka and Johansson (2010).

When we reviewed the dissertations, we saw a possible increase in subject-matter oriented research but also that studies focused on a particular subject matter tend to involve other didactical perspectives as well. This is also visible in the reports from the institutions. Small projects and occasional studies exist that are clearly classified as *stoffdidaktik* in the classical sense, like, for example, the work in Jönköping focusing on brackets (Gunnarsson, Sönnerhed, & Hernell, 2016). Kilhamn (2011) is also a clear example of a subject-matter focused study in which the theoretical and methodological perspectives are there to shed light on the students' understanding of the subject, not the other way around. For the bigger projects that have a subject-matter focus (groups of several people, typically working on a larger issue over several years, often with grants from the Swedish Research council), a closer look reveals that the empirical focus may not be the subject matter. It is difficult to discern any particular reasons for this. Perhaps this is in line with the Swedish tradition, which for some time has been rather weak in subject-matter oriented research, as remarked on by Björkqvist (2003). Perhaps Swedish research simply follows international trends, where subject-matter oriented research with a classical epistemological or psychological flavor has given way to socially oriented perspectives (Lerman, 2000). Examples include the already mentioned algebra project in Uppsala that relies heavily on textbook and curriculum analysis. The Foundational Number Sense project in Stockholm includes the construction of an ambitious framework for number sense which is clearly a subject-matter oriented endeavor. But the object of study is how parents and teachers in three different countries deal with number sense (Andrews & Sayers, 2015). A project in Örebro University has a focus on statistics but in strong conjunction with collaborative group work and specific theories for understanding such group work (Nilsson, Schindler, & Bakker, 2018). In Gothenburg, the VIDEOMAT project financed by the Nordic research council from 2011 to 2014 was aimed at finding hidden dimensions in algebra teaching through a comparative study of algebra classrooms in Sweden, Norway, Finland, and California, USA (Kilhamn & Röj-Lindberg, 2013). However, although this project was subject-matter oriented, the focus was just as much on the social features of the algebra classroom as on algebraic content.

The statistics project led by Nilsson also exemplifies another clear pattern among the projects reported to us that is not visible when reviewing PhD dissertations: A focus on design research. As we illustrate in our historical exposition, research related to the development of textbooks, tests, and other materials also existed in the eighties, but the interest for such research is surely renewed in Sweden. The above mentioned textbook project run by Ryve in Mälardalen University is one example. In Umeå University, Lithner runs the longest running research program in Sweden, with a focus on creative versus imitative reasoning (Lithner, 2008). For the last few years, this program has included a design research component focusing on understanding learning conditions for creative reasoning and their effects, as well as the cognitive and neurological basis (Jonsson et al., 2014). It is an interesting fact that two other design research efforts in Sweden also have neuroscience components. In a collaboration between the Klingberg cognitive neuroscience group at Karolinska Institutet and Helenius at the National Center for Mathematics Education, a free-to-use digital training program based on the number line has been developed and tested (Nemmi et. al, 2016). A local Gothenburg collaboration is currently testing another number-line based teaching sequence in grade 2 that, in turn, builds on previous design work at the NCM focusing on number sense for six-year-olds (Sterner & Helenius, 2015). Another unrelated design research effort based in Gothenburg in collaboration with Holgersson at University of Kristianstad also focuses on young children's number sense and finger-based interactions on ICT-based artefacts (Barendregt, Lindström, Rietz-Leppänen, Holgersson, & Ottosson, 2012). If we include research based on the learning study paradigm under the design umbrella, we would get an even longer list, including research from Karlstad University, Jönköping University, University of Gothenburg, and Stockholm University.

As noted above, different types of social perspectives have gained ground in Sweden. In a similar vein, sociopolitical perspectives are also growing stronger. In part, this has happened through influence from abroad. In Stockholm, Valero leads a group that has strong international presence in the sociopolitical field. This perspective was also strong in Malmö University, first with Wedege, later with Meaney, and now with Chronaki. All of these researchers are very active in the international scene for mathematics education and society (Chronaki, 2017).

Some other research foci that are shared between several university environments but that we only mention briefly are research on teacher education, with active groups in Kristianstad University, Linné University, and Karlstad University; and research on preschool mathematics, with research in Malmö University and Luleå University and an active group at NCM, as well as one group related to the learning study paradigm in Gothenburg, where Björklund runs a project focusing on early teaching of numbers.

Finally, one perspective that is mentioned by several university environments but that is not visible in our examination of PhD dissertations is the continuous professional development of teachers (CPD). Some projects study CPD without engaging in practical collabo-ration (Liljekvist, van Bommel, & Olin-Scheller, 2017), but in most cases, CPD is designed and in part carried out by the research group, for example, in Mälardalen University (Lindvall, 2017). Among these projects, we also find several quantitative studies, something which is generally not very common in Swedish research in mathematics education (Lindvall, Helenius, & Wiberg, 2017; Sterner, 2015).

HISTORICAL DEVELOPMENT IN FOUR PERIODS

As we indicated in the description by Schoenfeld in the introduction, research in mathematics education was initially carried out by researchers from other fields. It was not until the 1960s and 1970s that research in mathematics education became a discipline of its own, with professional organizations, journals, and so on. The Swedish development mimicked international development but took place at a later stage. To describe this development, we look at four distinct phases, 1950–1984, 1985–1999, 2000–2008, and, finally, 2009 to present.

1950-1984: Forerunners

This period can be characterized by the term *precursors*, or *forerunners*. Mathematics education was not yet a subject of its own, and there were no research groups with specific focus on mathematics teaching and learning. However, some of the traditions that still exist in Swedish educational research and also with importance for the mathematics education community were grounded here. Torsten Husén was one of the initiators of IEA, the International Association for Assessment of Educational Achievement (Husén, 1987). IEA is now responsible for international comparative studies like PIRLS (reading and language), TIMSS (mathematics and science) or TEDS-M (teacher education).

The work of IEA initiated an interest in international comparative studies that, for example, in Germany developed into a strong focus on quantitative research on teaching and instruction (cf Baumert et al., 2010 or chapter 6 in the National Presentation from PME-37 in Kiel, Bruder et al., 2013) with a clear focus in mathematics. In Sweden, this work split into one tradition that works with national tests and general TIMSS and PISA analysis (groups in Stockholm and Umea) and a separate tradition (in Gothenburg) that continues on the Husén path but seldom focuses on subject matter (with some exceptions; see, for example, Hansson, 2012). In recent years, the assessment research tradition has also taken two other paths. One is a focus on social or sociological aspects of assessment, and the other is on formative assessment. We will come back to a discussion of this.

It should be mentioned here that the testing tradition in Sweden also owes greatly to Fritz Wigforss (1886–1953), a national pioneer in the area of standardized testing. While Wigforss was active before 1950, his work was instrumental in the design of the comprehensive school system. From 1950–1962, a system for standardized testing based almost exclusively on Wigforss's work was designed and tested on a large scale alongside the development and expansion of the comprehensive school system itself. This work was led by Torsten Husén. In 1962, these tests became obligatory. Later, when the final examination in upper secondary school (*gymnasieskolan*) was gradually abolished, standardized yearly testing based on Wigforss's work was implemented there as well. In 1965, the responsibility for developing the standardized test moved between different institutions, and some of the central ideas from Wigforss eroded (Kilpatrick & Johansson, 1994). Bengt-Olov Ljung at the Stockholm Institute of Education played an important role in the development of standardized national tests for compulsory school (Ljung, 2000) and founded the PRIM-group in 1984. PRIM is still developing national tests in mathematics but is now located at Stockholm University.

In 1994, the standardized test was abolished in favor of a criterion-based system and new forms of national assessment not based on establishing norms (Kilpatrick & Johansson, 1994). The first Swedish publication in *Educational Studies in Mathematics* is, in some sense, in the tradition of Wigforss: combining issues of testing with discussions in the tradition of *stoffdidaktik* (Ekenstam & Nilsson, 1979).

Outside of the testing and assessment tradition, an exceptional body of research from this period, continuing into the present century, is the work of Magne, later often together with Engström (see Engström, 2016 for a review). This work was exceptional in both senses of the word by being both of good quality and unusual for the time in Sweden. Unfortunately, very little is available in English (one exception is Magne, 1978). Magne's work concerned what is now often termed low achievers in mathematics and started in the 1950s, coinciding with the start of compulsory schooling, which is unsurprising since comprehensive schooling in itself produces some of the problems Magne researched. The Magne tradition involves both the existence and characterization of mathematical difficulties, interventions, testing, and large-scale evaluations, all with a solid psychological grounding. This tradition is scarce in Swedish research, whereas our neighbor Finland houses very active and internationally respectable research in this field.

A project that, in some sense, signaled the structural changes in how mathematics education and research were organized in Sweden was the IMU project (*Individualiserad matematikundervisning*, Individualized mathematics instruction). The project was sponsored by the current National Board of Education with roughly 8 million euros in today's worth. Similar to the Individualized Prescribed Instruction (IPI) paradigm in the US (Erlwanger, 1973; Lindvall & Cox, 1970) IMU was a large-scale experiment with individualized instruction based on a self-study model. There are different views on the success of this experiment, and the project was abolished in the early 1970s.

The IMU project exemplifies an interesting breaking point in the history of Swedish school development because historically, changes in the school system were thoroughly tried out and evaluated, similar to the norm referenced testing and the comprehensive school system as such, IMU was rather a response to unintended effects of previous reforms from 1950 onward. The expansion of schooling caused a shortage of teachers and the comprehensiveness of the school (no tracking) led to a need for individualization inside the classroom (Marklund, 1985). It was assumed that pedagogical innovation would be the solution to these problems. IMU can, hence, be seen as an illustration of the end of the era of progressive systemic national school development but the start of an era of medium-scale developmental projects that still often relied on some form of governmental support.

Another important project from the years before 1985 was the PUMP project – Process analyses in Mathematics and Psycholinguistics. *Process* here should be taken to mean studies of teaching and learning situations. An accompanying part of the project was the development of diagnostic tests. While the testing tradition in Sweden was very strong at the time, PUMP represented something different by attempting to construct instruments that helped teachers understand what each student required. The PUMP project, led by Wiggo Kilborn, is one of very few coordinated research efforts with a clear focus on the teaching and learning of mathematics from this period. The project was mainly summarized in reports in Swedish but has clear influences from the work leading up to the revision of the national curriculum on mathematics in 1980 (Nämnaren, 1983). The tradition started by Kilborn is still alive and influential in Sweden, but despite the group working with PUMP being strong and having a clear research program, it never developed into a full research environment. We will come back to this issue later.

1985-1999: Establishing foundations

The years 1985–1999 can be seen as an era of establishing mathematics education research in Sweden as a subject in its own right, including establishing the term *matematikdidaktik*, or didactics of mathematics. This development followed an interesting path since it was not primarily driven by establishing particular lines of research. Using a metaphor from biology, the development in this era was, rather, about establishing an ecosystem where research can thrive. We chose the year 1985 as a breaking point because in the spring of 1985, media started to report on what was described as catastrophic results for the Swedish school in the IEA international comparative study SIMS. The Department of Education ordered an investigation that was published in 1986 involving, amongst other things, a critical inquiry of the quality and role of the mathematics textbooks and the quality of teacher education, as well as teaching in school (Emanuelsson, 2001; Utbildningsdepartementet, 1986).

The investigation sparked a lot of activity. Seminars in didactics of mathematics were started in several teacher educations. The professional development of upper secondary mathematics teachers was initiated. A group involving, among others, Barbro Grevholm was commissioned to review all the textbooks in mathematics (Grevholm, Nilsson, & Bratt, 1988) and the results of this investigation created a lot of further debate in national newspapers, TV, and radio. In 1986 around 30 persons active in the field of mathematics teacher education participated in a conference to discuss future possibilities (Emanuelsson, 2001). This can be seen as the first conference for mathematic education research and development in Sweden.

While these activities were important in themselves, a secondary effect that, in retrospect, seems just as important was that mathematics education as a phenomenon was given more attention in teacher education, in policy making, and in the general public mind. The increased focus on quality in mathematics teacher education inspired several teacher educators to write and present doctoral theses in the field, for example, Dahland at the University of Gothenburg, Möllehed at Malmö University, and Sandahl at Jönköping University (dissertation presented at Linköping University). These dissertations have a rather strong school focus, treating problem solving in school mathematics, ICT in school mathematics, and teachers' and pupils' views of mathematics, respectively (Dahland, 1998; Möllehed, 2001; Sandahl, 1997). Gradually, the literature used in teacher education was also replaced. Earlier literature was typically practice oriented in the methods tradition (in the US-terminology, sometimes called *didactical*, but not in the European sense of the word). An example of the new literature was *Didactics of mathematics - a Nordic perspective* (*Matematikdidaktik - ett nordiskt perspektiv*), edited by Grevholm (2001), in which the emerging Nordic tradition was presented by Nordic researchers.

A unique Swedish institution that became increasingly important in this era is *Matematikbiennalen*, a biennial national mathematics teacher conference. Established by Peder Claesson in 1980, the Biennial quickly became very popular among teachers and is a tradition that lives on today. In Björkqvist's (2003) overview of the state of mathematics education research in Sweden, it is noted that due to the Mathematics Biennial movement, the share of teachers who are regularly informed about modern research in mathematics education is probably higher in Sweden than in most countries. As research in mathematics education grew stronger, the content of the Biennial became more research oriented. In 1998, Grevholm suggested that a research conference should be arranged in conjunction with the Mathematics Biennial. This was the start of the Swedish Mathematics Education Research Seminar (MADIF) that, to date, has occurred 11 times. This was also when the Swedish Society for Research in Mathematics Education (SMDF) was established, coordinated at the start by Christer Bergsten and Barbro Grevholm (Bergsten, 2002).

At this point, a strong Nordic collaboration in mathematics education research was also initiated. The journal *NOMAD –Nordic Studies in Mathematics Education* was started in 1988 on the initiative of Göran Emanuelsson, who was working with Gunnar Gjone and Stieg Mellin-Olsen from Norway and Bengt Johansson from Sweden. The journal published its first issue in 1993 (Emanuelsson, 2001). About the same time, the first conference in the NORMA series (Nordic Research in Mathematics Education) was held in Lahti, Finland, and since then, the conference has been held every third or fourth year (Rönning, in preparation).

From the point of view of the historical development of mathematics education in Sweden, an important event was the creation of the journal *Nämnaren*, established in 1974. With Göran Emanuelsson as the founding editor, *Nämnaren* became not only an essential source of information and inspiration for mathematics teachers but also an arena for policy discussion and debate. Further, and particularly through the period dealt with here, *Nämnaren* acted as a focal point for several people in the Gothenburg area interested in mathematics education and with experience from textbook production, research, and teacher education, as well as curriculum studies and other policy related issues. Two members were Bengt Johansson and Wiggo Kilborn, who, supported by money from textbook sales, created and ran the International Seminar, which started after Johansson visited PME-8 in Sydney in 1984. Over the years, more than 100 top-level international researchers in mathematics education visited Gothenburg and often other parts of Sweden as well. During this period, Gothenburg and the *Nämnaren* group acted as the main hub for the dissemination of international research in mathematics education in Sweden (Emanuelsson & Mouwitz, 2012). At times, this influence had important policy implications. For example, the Swedish national curricula from 1994 for primary to upper secondary school were heavily influenced by the ideas leading up to the NCTM Standards (NCTM, 1989; 1991).

It is particularly interesting to study the Gothenburg environment because of its strong position in educational research in general in Sweden. When Strässer (2005) produced a list of PhDs in mathematics education up to 2004, he estimated that about a third of them originated from Gothenburg. One component of the Gothenburg environment at the time was the research revolving around Ference Marton and the phenomenographic approach (Marton, 1981). In 1987, Neuman had already presented her PhD containing studies of the number sense of children just before school entry. Neuman's work also included practically useful conceptualizations of early number sense (Neuman, 1987). Another dissertation in mathematics education from this group is Runesson's from 1999. Around this time, the group developed Variation theory, which later became the basis for Learning Studies, a teaching development method using a lesson study approach. The theories and methods emanating from this group can, then, be seen as some of the most influential in Swedish mathematics education and research to date. The perspective is still strong in Gothenburg, with, for example, Björklund working on a project on preschool mathematics supported by the Swedish Research Council (Björklund, 2018). Yet, despite the very large influence on policy, practice, and research, no coherent environment for mathematics education research has been created in Gothenburg to date, a somewhat puzzling fact that we will return to in our concluding section.

An important issue when studying the development of mathematics education research is its institutional belonging. The research emanating from the group around Marton is an example of when research on teaching and learning mathematics comes out of groups and institutions connected to general education. Another connection is obviously to mathematics. When Dunkels presented his PhD dissertation in Luleå 1996, it was the first dissertation concerning mathematics education that was presented at a mathematics institution (Dunkels, 1986). Other PhDs in mathematics education at this time were normally presented in educational institutions. In cases like Neuman (1987) or Runesson (1999) mentioned above, this was natural, since the work was carried out in an established research group within an educational institution. But the work of Dahland, Möllehed, and Sandahl or Bergsten (1990) and Hedrén (1990) at Linköping can be seen as more solitary, and the institutional belonging of such research is more unclear, although they were all presented in educational institutions. When research in mathematics education grew stronger, the competition for influence over the content of teacher education with mathematics itself grew stronger, particularly at universities with strong mathematics departments. It is in this perspective that Dunkels's dissertation is interesting.

The mathematics department at Umeå University also showed interest in mathematics education research. Mathematician and Head of Department Hans Wallin met Mogens Niss, who was a member of a committee evaluating the mathematics programs at Swedish universities. This was a few years before Niss was appointed a professor in mathematics education in Roskilde, but he was already an established name on the international mathematics education research arena. The result from this interaction can be seen as a precursor of the next era in Swedish mathematics education research. The idea was to establish a PhD program in mathematics education at the mathematics institution in Umeå. To help with the design of the program, an international committee with Niss, Gjone from Norway, and Björkqvist from Finland was formed. Johan Lithner, who had already completed a PhD in mathematics with Wallin as supervisor, completed a second PhD at Roskilde university with Niss as supervisor, further establishing this connection. The group received a grant from the National Agency for Higher Education to finance a few PhD students. Wallin argued that this new research direction at the department of mathematics should not compete with mathematicians for money. Lithner and Wallin supervised the PhD students, and Lithner effectively became the leader of the mathematics education research group in Umeå, which is now the most influential group in Swedish mathematics education research (Rönning, in preparation).

About as many PhD dissertations were published in 1985–2000 as in the whole period up to 1985. As remarked by Bergsten (2002), the period was also characterized by a large number of medium-scale school development projects typically founded by the National Agency of Education or its predecessor, the National Board of Education. As mentioned in the previous section in relation to IMU, this can be seen as a shift, where school development in mathematics was no longer driven by the state in a systematic way. Many of the development projects incorporated a research component, and several dissertations from this period are related to some of these projects (Bergsten, 2002).

Returning to the Gothenburg area, in 1999, after lobbying work by Emanuelsson and Johansson, the government decided to establish a National Center for Mathematics Education (NCM) in Gothenburg. Björkqvist (2003) remarked that the significance of this center is not just its activities but that its creation signaled political support for the area of mathematics education. NCM incorporated the *Nämnaren* group, along with many of the activities already going on in the Gothenburg area, and established a national reference library for mathematics education literature. NCM was also the host of several national investigations (Emanuelsson, 2001; Emanuelsson & Mouwitz, 2012)). When the center was created, both the mathematics department and the department of education were interested in housing it, but in the end, NCM was made a separate unit, not connected to a specific faculty. While this ensured independence, it also meant that the center was in some sense cut off from the local research community. Since the financing of the center did not include research, very little research could be done from within the center. When looking at this retrospectively, the influence of the Gothenburg environment over national professional development and school-related policy issues was probably strengthened, but the influence regarding mathematics education research may have been weakened.

In summary, one way of understanding this period as a whole is that it was initiated by a phenomenon that since has occurred in many countries: a "crisis" caused by the results of an international comparative study, the Second International Mathematics Study (Robitaille & Garden, 1989). While *mathematics* had previously been an important part of education, now *mathematics education* also became an object of interest in itself. This created a space for mathematics education to grow. The most influential individuals in this period did not primarily promote their research careers but the growth of the research field, the influence of this field locally and nationally, the establishment of international collaboration, and the development of institutions. Therefore, entering the next period, there is a structure for mathematics education research to grow on.

2000-2008: Establishing a field

Entering the new century, there was still no full professor in mathematics education in Sweden. The previous period had shown a rapid increase in the number of PhDs produced, but when Strässer (2005) made an overview of the Swedish research in mathematics education, he remarked that few researchers seemed to have published internationally, at least in top-level journals. All this changed rapidly.

The number-one defining event in this era was in 2000, when Riksbankens Jubileumsfond (RJ, the foundation of the Swedish Central bank) decided to finance a national graduate school in the didactics of mathematics comprising 15 graduate students. The decision was the outcome of an intense period of lobby work from many people (most of them mentioned in the previous chapter). Formally, the proposal came from the National Committee of Mathematics at the Royal Academy of Sciences. The goals of the graduate school were to develop the field of didactics of mathematics in Sweden and to supply teacher education in Sweden with more research-educated teacher educators. In some sense, similar goals had already been discussed in 1985, but now those goals could be formulated in terms of *strengthening mathematics education research*, rather than *strengthening mathematics education*. There was a strong infrastructure to build on. Some years earlier, SKM, the Swedish Committee for Mathematics Education, had been formed at the Royal Academy of Sciences. SKM was the Swedish contact point for ICMI. Members of SKM, like Gerd Brandell, Bengt Johansson, and others, had the necessary national and international contacts to help set up the graduate school. Hans Wallin from Umeå was instrumental in making the graduate school possible, and in Umeå, they had already pointed out a possible direction by placing their own graduate program under the wings of the mathematics department. It was generally considered that following this line would be a way to get mathematicians more involved in issues of mathematics education. The financing from RJ funded 15 PhD students, and more grants were then received from the Swedish Research Council, among others, so that in total, 24 students enrolled in the graduate school, which involved cooperation between 10 universities all over Sweden. While not all students finished with a PhD (a few stopped half way with licentiate exams, and some dropped out early or graduated in another subject), the number of PhDs awarded in Sweden was greatly affected by the RJ program. For example, in 2006 alone, 14 PhDs were awarded, which is the same number as for the whole period of 1985–1999 (Brandell, 2010; Rönning, in preparation).

The activities in the period of 2000–2008 were not limited to the RJ graduate school. Around one third of the PhDs awarded during the period were not from the graduate school. Of particular interest is the Umeå University environment, where PhDs not connected to the RJ graduate school during this period were awarded to seven people. Two were in the group for pedagogical measurement, but the others were supervised or co-supervised by Johan Lithner, in many cases with Hans Wallin, who invested 20 % of his time supervising students in mathematics education, despite not having any intentions of pursuing research in the area himself (Lithner, personal communication). In 2001, Tomas Bergqvist was the first person to earn his PhD in mathematics education from the mathematics department in Umeå. Bergqvist studied why students tend to practice remembering procedures instead of working conceptually with problems (Bergqvist, 2001). This is an almost timeless question in mathematics education, but in the Umeå environment, it has formed the basis of a 20-year-long research program that involves constructing frameworks for dealing with the question, researching tasks, textbooks, classrooms and teachers' views, designing and carrying out controlled experiments, and researching the psychological and neurological underpinnings of learning imitative and creative mathematical reasoning (Jonsson, Norqvist, Liljekvist, & Lithner, 2014).

One effect of the RJ graduate school was the formation of networks in Sweden, where different research environments could help each other with PhD supervision, giving courses in mathematics education research and so on. At the time, the number of senior researchers was small but increasing. The graduate school was, thus, not only a resource for producing PhDs but also a recourse for building competence among senior researchers. Moreover, collaborative efforts among participating universities also made it possible to invite international researchers to give courses or seminars or to act as discussants or opponents (Brandell, 2010; Lithner, personal communication).

Since the resources associated with the RJ school would eventually run out, there was a need to create another structure to support collaboration and exchange. Barbro Grevholm, at this point a professor at the University of Agder in Norway, managed to find funding to set up NoGSME, the Nordic Graduate School in Mathematics Education, which ran from 2004–2008. NoGSME ran PhD courses and seminars and also arranged seminars for supervisors. Since so few senior researchers existed in the field at the beginning of this period, NoGSME played an important role in building up supervising competence (Grevholm, 2009).

2009-present

Over the course of the previous three periods, the number of PhDs per year grew in an exponential manner. In the period leading up to the present, development was boosted by the RJ graduate school. So how did research in mathematics education in Sweden progress after 2008, when almost all dissertations resulting from the RJ school had already been published? In fact, the PhD volume has continued to rise, just not as fast as before. From 2009 to 2017, around 75 dissertations were published. Several of these were supervised or co-supervised by researchers from the RJ graduation school generation. This can be seen as a maturation of the field. The number of professors has risen substantially. It took until 2001 for Sweden to install the first professor in mathematics education research: Rudolf Strässer at Luleå. There are now 12 professors, four of which have backgrounds in the RJ graduate school, and one with a Swedish PhD from the same period. The rest have PhDs from other countries and have come here to serve as professors. Despite the fact that over 100 people have earned PhDs in mathematics education since 2000, most of whom are still active in the field, the demand for lecturers and professors in mathematics education is bigger than the supply. Will this demand continue to be a driver?

Strässer (2005) identified a lack of participation on the international arena, exemplified by a very sparse participation at PME conferences from 2000 to 2004 and very few publications by Swedish researchers in the top international journals.

This has changed. For example, in each of the last six years, researchers working in Sweden have published in *Educational Studies in Mathematics*, and Swedish researchers are now regularly attending PME as well as other major conferences in the field. The dissertations that are not monographs now regularly contain articles published in international journals. At the level of individual researchers and their activities, Sweden has taken a large leap.

One objective with the RJ graduate school was to establish research in mathematics education at more universities and, in particular, in mathematics departments. The activities and trends in the involved departments in 2009 were discussed by Brandell (2010). She considered three groups: Umeå University and Luleå University, with stable activities even before RJ, when the activities had developed further; universities at Kristianstad, Linköping, and Linné, which were new environments at the RJ start but that still had activities afterward; and, finally, the University of Gothenburg, Stockholm University, and Uppsala University, where very little activity could be seen two years after the end of the RJ project. The five environments in the first two groups all had senior researchers, doctoral students, and active research groups. The expectation was that departments with active research environments with senior researchers and doctoral students would be able to develop further. Looking at the situation now, a decade later, it is clear that it was not easy to predict the future. Linné University, Linköping University, and Kristianstad University still have active groups. Individual researchers continue to publish and often develop their research along new lines. The same goes for Luleå University. The environments, as such, do not seem to have developed consistent research programs. Although the number of researchers grows, they do not seem to consolidate into larger research groups. According to Bergsten (2010), the Swedish community is characterized by groups of researchers and individuals, working in isolation within different theoretical orientations. Examples of cross university collaboration exist though. A group with members from Malmö, Luleå, NCM, and Bergen (Norway) has published relatively extensively on preschool mathematics education (Helenius et al. 2016). There is also collaboration between Linné University and the University of Gothenburg in the preschool area (Palmér & Björklund, 2016). A group with members at Linné University, Karlstad University, and Luleå University work with entrepreneurial learning with a grant from the Kamprad foundation (Palmér, Johansson, & Karlsson, 2018). It seems to be easier for researchers in Sweden to unite around content and collaborate across universities than to maintain a local research group working with a coherent research program.

Uppsala is an interesting case, as it was one of the environments where activities were rather sparse right after the RJ graduate school (Brandell, 2010). However, now there is a rather coherent group there working on a project on school algebra financed by the research council, with Kirsti Hemmi, based in Finland, and Uffe Tomas Jankvist, from Denmark, working as visiting professors.

Stockholm University is another environment that did not flourish after the RJ-project, but a few years ago, the Teacher Education University College and the university merged, and a substantial restructuring of the research environment followed. A rather large Department of Mathematics and Science Didactics (MND) was formed, also incorporating the activities of the PRIM group working with assessment. Using strategic resources, two professors from abroad were hired, Paul Andrews and Paola

Valero, both firmly established in their respective fields. Other senior researchers have been recruited from both within Sweden and abroad. There is a consistent flow of doctoral students finishing their work, many from the local environment. There is also an active master's program and an associated set of courses, as well as doctoral courses. While the mathematics department is still not very involved, the mathematics education community at MND has certainly grown to become one of the centers of mathematics education research in Sweden.

Another interesting case is Mälardalen University, where Ryve coordinates several large-scale projects. Mälardalen was not part of the RJ school, but a combination of strategic resource allocation from the university, long-running Research Council funding, large-scale collaboration with the municipality, and an ambitious textbook project means that the research environment is now one of the strongest in Sweden. There is also an additional Research Council-funded project headed by a postdoc, Henrik van Steenbrugge.

The most stable environment in Sweden is, however, still the one at Umeå University. In 2006, the Umeå Mathematics Education Research Centre (UMERC) was founded by the vice-chancellor of Umeå University with Lithner as a leader. From the outset, UMERC housed different projects, but the most important was the project headed by Lithner, in which the previous PhD students Ewa Bergqvist, Tomas Bergqvist, Jesper Boesen, and Torulf Palm were also members. The interesting development since 2009 is that both Palm and Ewa Bergqvist, together with Magnus Österholm, have formed separate groups of their own, pursuing different projects (Andersson & Palm, 2017; Bergqvist, Theens, & Österholm, 2016). Including Lithner's group, now combining the original ideas with psychology and neuroscience, and a group with a special education focus, there are now four groups under the UMERC umbrella, each with senior researchers and doctoral students.

CONCLUSIONS AND LOOKING AHEAD

Since the "explosion" from 2000 and onward, mathematics education research in Sweden has become a strong subject with a healthy amount of active researchers distributed over most universities. However, in many places, the research is carried out by one or only a few researchers, with little resources and coordination. Some organize themselves in groups across universities, sometimes with international collaboration. There are few places that have managed to establish stable research environments that carry out systematic research programs, the prime example being Umeå University. Mälardalen and Stockholm Universities now have several strong groups working in different areas. Coordinated research environments that do exist seem to always be the result of strategic investments from the local university in combination with the existence of (or recruitment of) strong research leaders.

Also notable is that one of the largest universities, Gothenburg, which houses a strong tradition in mathematics education, was the birthplace of NOMAD, and is the location of the National Center for Mathematics Education, has no professor in mathematics education. Bengt Johansson was a professor in mathematics education for a short time before his retirement (Emanuelsson & Mouwitz, 2012). Although, at times, many strong groups in general education attract researchers with an interest in mathematics education, particularly the Variation theory group, no consistent research programs focusing on mathematics education research exist. Yet, the University of Gothenburg still continues to produce many PhDs. An examination of the list of PhDs in mathematics education shows that during the last ten years, 14 of the 77 dissertations produced have come from the University of Gothenburg, more than from any other university.

Perhaps Gothenburg can serve as an example to illustrate that unless the local university sees mathematics education as a strategic subject, the subject will remain relatively fragmented. A potential explanation for this is related to national research funding. Typically, it is difficult to obtain funding that supports more than the equivalence of one researcher full time for four years. This seems to stimulate the creation of small research groups working on short-term projects rather than research programs that are run in larger groups that are stable over time. Once a strong group is established, with the help of local funding, the national funding system seems to work well for supporting such groups. But the same funding system is perhaps not working very well to support the *establishment* of such groups. UMERC in Umeå is the strongest group in Swedish mathematics education research. It has brought in several grants from the national founding organizations, but about half of its financing has been local funding (Lithner, personal communication). While most of these resources have been obtained in local competitions, such funding is still more predictable than having to rely on external funding. Umeå University has shown a belief in mathematics education research, which has paid off.

Looking at the content of the research, we have to agree with Bergsten (2010). The Swedish research community is quite hard to characterize. Bergsten in particular noted that Swedish researchers seemed to work in isolation with different theoretical orientations. We have not so much focused on theory but, rather, the types of questions that are dealt with. This perhaps makes it a little easier to distinguish some trends. Our review of PhD dissertations, for example, shows that issues related to communication and reasoning have grown strong during the last decade, but it is not clear if this is just following an international trend or if it is related to some particular local issues. As we mentioned, the different researchers throughout Sweden dealing with communication and reasoning are not united by a shared theoretical approach. A similar situation holds for textbook research. It is a comparatively common theme in Swedish research, but the theoretical orientations vary greatly. The only Swedish specialty might be studies that build on Variation theory and other phenomenographic approaches.

In summary, mathematics education research in Sweden has a rather long tradition if research formally conducted within other subjects but actually dealing with the teaching and learning of mathematics is included. There is also a relatively old tradition of international collaboration. However, it took a long time for mathematics education research to become an established Swedish research area in its own right. This late development is an important background variable when viewing the current situation. Over the last decades, though, research in mathematics education in Sweden has developed rapidly. It is our hope that already established research groups will continue to consolidate their positions, and hopefully we will see other groups develop and become viable, contributing to a sustainable environment for useful research on the teaching and learning of mathematics.

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INDEX

INDEX OF AUTHORS AND CO-AUTHORS (ALL VOLUMES)

A

Aaten, Aaltje Berendina ....5 - 3
Aberin, Maria Alva......................5 - 49
Abo-Mokh, Dania Jamal ....2 - 139
Abtahi, Yasmine ....1 - 95
Abu-Naja, Muhammad... 5 - 60, 5 - 201
Ada, Tuba ....5 - 289
Adler, Jill ....1 - 95
Ahl, Linda Marie ....2 - 3
Ahmadpour Mobarakeh, Fatemeh.........
....2 - 67
Akahane, Kosuke.......................5 - 202
Akın, Ayça...................................5 - 4
Aksela, Maija Katariina.............5 - 144
Aksoy, Esra...................... 5 - 5, 5 - 128
Aksoy, Mehmet Akif.................5 - 128
Alamolhodaei, Hassan.................5 - 25
Alapala, Burcu ....5 - 6
Albano, Giovannina.......... 5 - 7, 5 - 203
Albarracín, Lluís........................5 - 237
Alcock, Lara ....3 - 139
Alderton, Julie ....2 - 11
Ali, Clement Ayarebilla ....5 - 204
Ali, Sikunder................................5 - 8
Alibali, Martha W......................3 - 427
Alvarez, James A. Mendoza.......5 - 147
Amador, Julie .... 2 - 19, 5 - 191
Ambrus, Gabriella ....5 - 9
Amresh, Ashish ....4 - 459
Anabousy, Ahlam Adnan ....
.... 2 - 27, 2 - 139
Anastasakis, Marinos...................5 - 62
Anderson, Hans Kweku.............5 - 204
Anderson, Judy Anne ..... 1 - 187, 2 - 35
Andrà, Chiara ....1 - 189
Andresen, Mette ....2 - 43
Andrew, Izsak............................5 - 205
Andrews, Nick .... 2 - 51, 5 - 73
Applebaum, Mark........................5 - 10
Appova, Aina..............................2 - 59
Aqazade, Mahtob ....5 - 209
Arabaci, Nil ....3 - 203
Arai, Mitsue.................................5 - 11
Arbona, Eva.................................5 - 12
Argun, Ziya ....2 - 403
Arteaga, Pedro..............2 - 131, 5 - 232
Asami-Johansson, Yukiko ....5 - 13
Asghari, Amir Hossein................2 - 67
Asgharzadeh, Nasrin ....5 - 309
Askew, Mike .... 1 - 125, 2 - 75, 5 - 57
Assmus, Daniela..........................5 - 14
Ataide Pinheiro, Weverton..........5 - 15
Athanasopoulou, Anna................4 - 91
Auquière, Amélie ....5 - 36
Awoniyi, Florence Christianah ....2 - 83
Ayala-Altamirano, Cristina..........5 - 16
Ayalon, Michal.................2 - 91, 4 - 51
Ayber, Gözde ....5 - 170

B

Baba, Takuya...............................5 - 59
Baccaglini-Frank, Anna ..2 - 99, 3 - 387
Baek, Younghyun......................5 - 305
Bagota, Mónika ....5 - 169
Bakogianni, Dionysia................4 - 291
Bakos, Sandy.............................2 - 107
Baldry, Fay.......................5 - 17, 5 - 47
Bampili, Amalia-Christina ....2 - 115
Baor, Rinat ....5 - 63
Baram, Ami ....4 - 139
Barkai, Ruthi ....1 - 125, 4 - 275
Barmby, Patrick.........................2 - 243
Barwell, Richard ....2 - 123
Batanero, Carmen......................2 - 131
Batista, Rita ....5 - 24
Baya'a, Nimer Fayez ....2 - 139
Baye, Ariane...............................5 - 139
Beckmann, Sybilla ....3 - 155, 5 - 205
Beltrán-Meneu, María José ....5 - 12
Bempeni, Maria.........................2 - 147
Benedicto Baldonado, Clara ....5 - 18

Berger, Margot ....5 - 19
Berggren, Johannes.....................5 - 206
Berman, Abraham.......................4 - 435
Bernabeu, Melania......................2 - 155
Bertin, Charlotte ....1 - 155
Beswick, Kim ....1 - 3
Bikner-Ahsbahs, Angelika ....5 - 20
Billion, Lara................................5 - 207
Biton, Yaniv ....5 - 21
Biza, Irene.................... 3 - 195, 5 - 245
Björklund, Camilla ....
..........................1 - 125, 3 - 267, 5 - 42
Bjorkman, Katie ....5 - 22
Blanco, Teresa F.........................5 - 208
Böckmann, Matthias...................2 - 163
Bofferding, Laura .... 5 - 23, 5 - 209
Borba, Rute Elizabete..................5 - 24
Borji, Vahid .... 5 - 25, 5 - 154
Borromeo Ferri, Rita ...... 3 - 91, 3 - 459
Bozkurt, Gulay ....5 - 26
Bresock, Krista Kay....................2 - 171
Brkovic, Irma..............................3 - 267
Bronkhorst, Hugo ....5 - 27
Brown, Julian Tom ....2 - 179
Brown, Rachael .... 5 - 125, 5 - 276
Brunner, Esther...........................3 - 315
Bruno, Gustavo...........................1 - 203
Buforn, Àngela ....2 - 187
Bukova Güzel, Esra...... 5 - 127, 5 - 278
Büscher, Christian ....2 - 195
Buzaglo, Meir...............................3 - 11

C

Cakiroglu, Erdinc ....5 - 224
Callejo, M Luz............................4 - 107
Callos Cronberg, Florenda ....5 - 123
Camacho Machín, Matías.............5 - 61
Campbell, Anita..........................5 - 210
Canavarro, Ana Paula.................2 - 203
Cao, Yiming ....4 - 219, 5 - 211, 5 - 282
Caponi, Matteo ....5 - 212
Carney, Michele .... 2 - 211, 5 - 213
Carotenuto, Gemma...... 1 - 247, 2 - 219
Carrión Vázquez, Vicente ....4 - 3
Carroll, David.............................2 - 395
Caseiro, Ana Sofia ....5 - 155
Castro-Rodríguez, Elena ....3 - 465
Castro, Enrique...........................3 - 465
Cavalcante, Alexandre ....
.... 5 - 214, 5 - 281, 5 - 288
Cavalcanti, Érica ....5 - 215
Cavey, Laurie ....5 - 213
Cavka, Sara ....5 - 216
Cawley, Anne ....2 - 227
Cedro, Wellington Lima ....5 - 28
Ceulemans, Eva...........................4 - 339
Champagne, Kevin.....................1 - 155
Chan, Man Ching Esther.......................
......................................1 - 223, 1 - 225
Chandia, Eugenio ....5 - 286
Charneca, Eliane ....5 - 221
Chen, Chang-Hua.........................5 - 29
Chen, Chia Huang ....5 - 217
Chen, Ching-Shu ....5 - 218
Chen, Donglin ....5 - 219
Chen, Hanjun..............................5 - 188
Chen, Jian-Cheng ....5 - 30, 5 - 252
Chen, Jun ....5 - 188
Chen, Qian....................................5 - 67
Chen, Ya-Wen ....5 - 252
Cheng, Chun-Yung ....5 - 265
Cheng, Karen..............................5 - 250
Cheng, Ying-Hao ....5 - 30, 5 - 252
Cheung, Kwok-cheung.........................
.... 5 - 31, 5 - 268, 5 - 290
Chidthachack, Sousada ....5 - 32
Chimoni, Maria ....2 - 235
Ching, Yu-hui.............................5 - 305
Chirinda, Brantina ....2 - 243
Chittoori, Bharskar ....5 - 305
Cho, Hyungmi ....5 - 86
Choi-Koh, Sang Sook...................4 - 83
Choi, InYong................................5 - 93
Chorney, Sean ....2 - 251
Choy, Ban Heng ....3 - 147

Christiansen, Iben Maj ....1 - 211, 2 - 259, 3 - 443
Christou, Konstantinos ....2 - 267
Chu, Ting-hua.............. 5 - 253, 5 - 308
Chumachenko, Dmitry ....5 - 185
Cifarelli, Victor...........................5 - 33
Clarke, David............... 1 - 225, 3 - 379
Clements, Douglas.....................1 - 125
Coles, Alf..................... 1 - 199, 2 - 275
Colipan, Ximena........................5 - 220
Conner, AnnaMarie...................2 - 283
Connolly, Mary..........................2 - 291
Cooper, Jason .... 2 - 299, 2 - 307
Coppola, Cristina.......................2 - 219
Costa, Conceição .... 5 - 221, 5 - 225
Costa, Maria Cristina...................5 - 34
Crisan, Cosette...........................2 - 315
Crowder, Anita Louise ....5 - 222
Csíkos, Csaba .... 5 - 9, 5 - 169
Culligan, Karla .... 5 - 35, 5 - 186
Cusi, Annalisa.............. 1 - 191, 2 - 323

D

Daher, Wajeeh .... 2 - 139, 5 - 54
Dahl, Bettina...............................2 - 43
Dartnell, Pablo .... 5 - 50, 5 - 236
Datsogianni, Anastasia ....2 - 331
Davis, Alan.................. 3 - 163, 4 - 323
Davis, Brent ....1 - 155
de Chambrier, Anne-Françoise............. ....5 - 36, 5 - 174, 5 - 223
De Smedt, Bert ....5 - 175
de Vries, Siebrich ....5 - 151
Degrande, Tine .... 4 - 339, 5 - 37
Dello Iacono, Umberto ....5 - 203
Demonty, Isabelle......................4 - 371
Deprez, Johan ....5 - 3
Di Bernardo, Rosa ....1 - 247
Di Martino, Pietro................................ ....1 - 189, 2 - 99, 2 - 219, 2 - 339
Dias, Thierry..............................1 - 207
Dicks, Joseph.............................5 - 186
Diego, José Manuel ....5 - 208
Dierenonck, Christophe ..5 - 36, 5 - 174
Diez-Palomar, Javier.....1 - 233, 5 - 303
Dilberoglu, Merve........2 - 347, 5 - 224
Dinçer, Bahar ....5 - 38
Ding, Liping ....2 - 355
Dogan, Oguzhan........................3 - 203
Domingos, António .... 5 - 34, 5 - 164, 5 - 225
Dong, Lianchun..............5 - 39, 5 - 226
Dörfler, Tobias ....5 - 160
Dorneles, Beatriz..........5 - 119, 5 - 273
Downton, Ann ....5 - 227
Dreher, Anika.. 1 - 219, 2 - 363, 5 - 228
Dreyøe, Jonas ....2 - 371
Drijvers, Paul.............................1 - 193
Duranczyk, Irene ....2 - 227
Dyrvold, Anneli.........................2 - 379

E

Eames, Cheryl Lynn..................2 - 387
Earnest, Darrell ... 2 - 19, 5 - 40, 5 - 229
Edmonds-Wathen, Cris ....5 - 41
Edwards, Antony.......................2 - 395
Eichler, Andreas ....5 - 160
Ekdahl, Anna-Lena ....5 - 42
EL Mouhayar, Rabih Raif...........5 - 43
Elicer, Raimundo.......................1 - 203
Emre-Akdogan, Elcin................2 - 403
Engelmann, Laura ....5 - 141
Eraikhuemen, Lucy ....5 - 230
Erath, Kirstin ....5 - 231
Ercire, Yusuf Emre.........5 - 44, 5 - 128
Eriksson, Helena ....1 - 255
Essien, Anthony A. ....5 - 45
Evans, Tanya ....5 - 46
Even, Ruhama ....3 - 99
Eylon, Bat-Sheva.......................4 - 139
Ezeife, Anthony N.....................5 - 133

F

Fagnant, Annick ....5 - 36, 5 - 174
Farmer, Sherri ....5 - 209
Farrugía Fuentes, Mauricio ....4 - 3

Fellus, Osnat......1 - 263, 2 - 411, 5 - 21
Fendi, Aysun..............................5 - 171
Fernández-León, Aurora ....3 - 355
Fernández, Ceneida ....
........................2 - 187, 2 - 459, 3 - 147
Ferrara, Francesca......... 1 - 193, 2 - 419
Ferrari, Giulia ....2 - 419
Ferrari, Pier Luigi ....5 - 7
Fidje, Anders Støle ....2 - 427
Fiorentini, Dario .... 3 - 331, 5 - 267
Flodén, Liselott............................5 - 77
Font, Vicenç .... 5 - 25, 5 - 154
Förster, Frank ....5 - 14
Forsythe, Susan Kathleen ....5 - 47
Foxworthy, Julianne Elsie ....5 - 48
Francisco, John ....5 - 53
Freiman, Viktor ....5 - 10
Friesen, Marita............. 2 - 435, 3 - 275
Frisk, Susanne............................5 - 123
Fritzlar, Torsten .... 2 - 443, 5 - 14
Fujita, Taro ....3 - 227

G

Garcia Moreno-Esteva, Enrique ....
.... 1 - 209, 2 - 475
Garcia, Mark Lester......................5 - 49
Gardes, Marie-Line.....................1 - 207
Gardner, Amber ....5 - 78
Gartland, Sara ....5 - 65
Gavilán-Izquierdo, José María ...3 - 355
Gea, María Magdalena . 2 - 131, 5 - 232
Gebel, Inga .... 5 - 233, 5 - 234
Geisler, Sebastian ....2 - 451
Gélinas, Marie-Sophie................1 - 263
Giauque, Nadine............ 5 - 36, 5 - 139
Gifford, Sue ....5 - 173
Gil Cuadra, Francisco.................5 - 136
Giménez, Joaquim ....5 - 303
Glanfield, Florence.....................1 - 155
Glasnović Gracin, Dubravka ......3 - 283
Godfrey, Alison Mary ....5 - 235
Goedhart, Martin .... 5 - 3, 5 - 27
Golding, Jennie..........................4 - 211
Gomez, David Maximiliano.................
.... 1 - 219, 5 - 50, 5 - 236
Gonzales, Alicia ....5 - 40
González-Forte, Juan Manuel ....2 - 459
González-Regaña, Alfonso J......3 - 355
Goral, Johanna..... 5 - 51, 5 - 94, 5 - 283
Gorgal, Alejandro.......................5 - 208
Gorgorió, Núria ....5 - 237
Graven, Mellony ....4 - 331
Gregorio, Francesca ....2 - 339
Griese, Birgit ....2 - 467
Grüßing, Maike ....3 - 315
Gucler, Beste ....2 - 403
Guillemette, David ....1 - 95
Guimarães, Gilda Lisbôa...........5 - 215
Gun, Ozge..................................5 - 256
Gunnarsson, Robert............................
.... 3 - 451, 5 - 55, 5 - 238
Günther, Claudia-Susanne ....5 - 239
Guo, Kan ....4 - 219
Gurjanow, Iwan...........................5 - 52
Gustavsen, Trond Stølen ....5 - 8
Gutiérrez-Fallas, Luis Fabián.....5 - 247
Gutierrez, Angel................5 - 12, 5 - 18

H

Haataja, Eeva................1 - 209, 2 - 475
Hähkiöniemi, Markus......4 - 307, 5 - 53
Haj Yahya, Aehsan.......................5 - 54
Haja-Becker, Shajahan ....5 - 240
Hakamata, Ryoto..........................3 - 43
Håkansson, Per ....5 - 55
Hals, Sigurd ....5 - 8
Hamanaka, Hiroaki ....5 - 157, 5 - 241
Hamm, Jill ....3 - 3, 3 - 51
Hamo, Pircha ....3 - 11
Hanazono, Hayato ....5 - 242
Hanke, Erik ....5 - 20
Hannula-Sormunen, Minna ....1 - 125
Hannula, Jani ....5 - 56
Hannula, Markku................................
1 - 19, 1 - 195, 1 - 223, 2 - 475, 5 - 167
Hansa, Sameera ....5 - 57

Hansson, Lena ....5 - 243
Hansson, Örjan .... 5 - 243, 5 - 244
Harbison, Lorraine......................5 - 301
Häsä, Jokke..................... 4 - 299, 5 - 58
Haser, Çiğdem ...2 - 347, 3 - 19, 5 - 224
Hassidov, Dina ....5 - 72
Hattermann, Mathias ....3 - 27
Hattori, Yuichiro..........................5 - 59
Hawks, Michelle Christine ....3 - 35
Hayashi, Tomoko ....5 - 258
Hayata, Toru ....3 - 43
Healy, Lulu ....5 - 245
Heck, Daniel .... 3 - 3, 3 - 51
Heidtmann, Pia ....5 - 77
Heil, Cathleen ....3 - 59
Heinrich, Daniel ....3 - 27
Heinze, Aiso ....
.............2 - 363, 3 - 67, 4 - 179, 4 - 379
Helenius, Ola .... 1 - 273, 2 - 3
Helf, Nader .... 5 - 60, 5 - 201
Helliwell, Tracy ....3 - 75
Hemmi, Kirsti ....
.........................3 - 259, 4 - 203, 5 - 150
Hendrikse, Petra ....5 - 246
Henriques, Ana Cláudia Correia
Batalha .... 5 - 247, 5 - 280
Herendiné Kónya, Eszter...............5 - 9
Herheim, Rune............... 1 - 95, 4 - 227
Hernández Hernández, Alexánder.........
....5 - 61
Héroux, Sabrina..........................5 - 248
Hertling, Dorothea......................5 - 249
Hewitt, Dave ....5 - 62
Heyd-Metzuyanim, Einat ....
...........................4 - 155, 4 - 427, 5 - 63
Hilton, Caroline Ann ....3 - 83
Hiltunn, Jenna..............................5 - 53
Hoch, Stefan ....5 - 64
Hock, Natalie...............................3 - 91
Hodge, Lynn ....5 - 250
Hodkowski, Nicola M ....3 - 163
Hoffman, Abigail...........................3 - 3
Hoffmann, Anna ....3 - 99
Hofmann, Andrea ....5 - 8
Hohensee, Charles........................5 - 65
Holm, Jennifer................5 - 66, 5 - 251
Holmberg, Britt ....5 - 123
Holzäpfel, Lars ....5 - 228
Hoover, Pippa ....3 - 51
Horne, Marj ....4 - 131
Hotomski, Mirjana ....3 - 107
Höveler, Karina ....3 - 115
Howard, Emma ....3 - 371
Høynes, Siri-Malén ....3 - 123
Hreinsdóttir, Freyja ....5 - 92
Hsieh, Chia-Chun ....5 - 69
Hsieh, Feng-Jui ....5 - 67
Hsieh, Shu-Chi ....5 - 298
Hsu, Hui-Yu ....5 - 30, 5 - 252
Huang, Chih Hsien ....5 - 68
Huang, Hsin-Mei..............3 - 67, 5 - 69
Huang, Hui-chuan ....5 - 253, 5 - 308
Hubeňáková, Veronika.................5 - 70
Hughes, Elizabeth ....5 - 162
Huitzilopochtli, Salvador ....5 - 71

I

Iannone, Paola ....3 - 131
Ieong, Man-kai .. 5 - 31, 5 - 268, 5 - 290
Ilany, Bat-Sheva...............3 - 11, 5 - 72
Inglis, Matthew ....
.... 1 - 195, 3 - 139, 4 - 451
Ingram, Jenni....................2 - 51, 5 - 73
Innabi, Hanan ....5 - 74
Isler-Baykal, Isil ....5 - 6
Ivars, Pedro ....3 - 147
Izsak, Andrew Gyula..................3 - 155

J

Jaime, Adela ....5 - 18, 5 - 284
Jakobsen, Arne ....4 - 35
Jans, Véronique ....5 - 139
Javaherpour, Azadeh ....5 - 214
Jaworski, Barbara ....5 - 62
Jeschke, Colin ....5 - 75
Jiang, Chunlian...........................5 - 254

Jiang, Zheng ....5 - 76
Johansson, Helena ....5 - 77
Johnson, Heather Lynn..........................
..........................3 - 163, 4 - 323, 5 - 78
Jojo, Zingiswa Mybert................5 - 255
Jokiranta, Kaisa ....5 - 53
Joklitschke, Julia.........................3 - 171
Jones, Keith .... 2 - 355, 3 - 227
Jorgensen, Cody ....3 - 163
Jorgensen, Theresa A. ....5 - 147
Jung, Hyunyi................................5 - 79
Juter, Kristina .... 5 - 243, 5 - 244

K

Kaarstein, Hege .... 1 - 197, 3 - 179
Kaas, Thomas ....5 - 80
Kabael, Tangül ....5 - 88
Kageyama, Kazuya.....................3 - 187
Kajander, Ann...............................5 - 66
Kalaycı, Ozlem ....5 - 256
Kan, Guo.....................................1 - 197
Kang, Hyun Young.......................5 - 93
Kang, Soonja ....5 - 257
Kang, Yueyuan ....5 - 188
Karpinski-Siebold, Nadja ....2 - 443
Karsenty, Ronnie .... 1 - 199, 4 - 123
Kaur, Berinderjeet ....5 - 81
Kawasaki, Masato.......................5 - 258
Kawazoe, Mitsuru.........................5 - 82
Kayali, Lina ....3 - 195
Kazemi, Maryam ....5 - 309
Kennedy, Nadia Stoyanova ....5 - 83
Kerger, Sylvie.............................5 - 139
Ki, Wing Wah.............................5 - 261
Kilhamn, Cecilia..................................
.............1 - 273, 5 - 84, 5 - 123, 5 - 259
Kilic, Hulya ....3 - 203
Kim, Do-Hong ....5 - 254
Kim, Doyen ....5 - 85
Kim, Hee-jeong .... 3 - 211, 5 - 86
Kim, Jin-Ho ....4 - 147
Kim, Jiwon ....5 - 257
Kim, Nick ....5 - 250
Kim, Ok-Kyeong.......................3 - 219
Kim, Yeon ....5 - 120
Kimai, Patrick ....2 - 227
King, Shande ....5 - 250
Kızıltoprak, Ayhan ....5 - 87
Kızıltoprak, Fatma........................5 - 88
Klemp, Torunn ....3 - 123
Klöpping, Peter Michael ....5 - 89
Klunter, Martina........................3 - 283
Klymchuk, Sergiy ....5 - 46
Knutsson, Lena..........................5 - 123
Koichu, Boris ....3 - 347, 4 - 435
Komatsu, Kotaro ....3 - 227
Kontorovich, Igor' ....3 - 235
Kortenkamp, Ulrich..............................
.... 5 - 51, 5 - 94, 5 - 283
Kose, Nilufer Yavuzsoy ....
.... 5 - 87, 5 - 180, 5 - 196
Kourouniotis, Christos ....5 - 110
Kovács, Zoltán ....5 - 9
Kozaklı Ülger, Tuğçe ....5 - 90
Krause, Christina M. ....5 - 91
Krawitz, Janina..........................3 - 243
Kriesler, Alice ....2 - 35
Kristinsdóttir, Bjarnheiður (Bea) ....
....5 - 92
Kristmanson, Paula ....5 - 186
Krummenauer, Jens.......3 - 251, 5 - 260
Krzywacki, Heidi ....3 - 259
Ku, Na-Young ....5 - 93
Kuhn, Christiane ....5 - 75
Kullberg, Angelika........1 - 125, 3 - 267
Kunimune, Susumu ....5 - 157
Kuntze, Sebastian ....
...........2 - 435, 3 - 251, 3 - 275, 5 - 260
Kuzle, Ana..
.............3 - 283, 5 - 89, 5 - 233, 5 - 234
Kwon, Oh Nam ....5 - 85, 5 - 100

L

L'Italien-Bruneau, Rox-Anne .... 1 - 155
Ladel, Silke ....5 - 94
Lagrange, Jean-baptiste.............3 - 291

Lahdenperä, Juulia Josefiina ....5 - 95
Laine, Anu ....5 - 237
Lake, Elizabeth ....5 - 96
Lambert, Kerrilyn ....3 - 3
Larsen, Dorte Moeskær ....2 - 371
Larson, Niclas ....5 - 97
Larsson, Kerstin ....5 - 98
Laster, Lori Ann ....5 - 161
Lau, Wing Fat ....5 - 99
Lavicza, Zsolt ....5 - 92
Lee, Arthur Man Sang ....5 - 261
Lee, Kyeong-Hwa ....5 - 112
Lee, Kyungwon ....5 - 100
Lee, Shin-Yi .... 5 - 101, 5 - 262
Lehavi, Yaron ....4 - 139
Lehesvuori, Sami ....5 - 53
Lehner, Matthias C. ....5 - 102
Lei, Kin Hang ....5 - 15
Leikin, Mark ....4 - 403
Leikin, Roza ....4 - 403
Lem, Stephanie ....3 - 299
Leong, Swee Ling ....5 - 103
Leppinen, Jussi ....5 - 58
Lerman, Steve ....1 - 95
Leshota, Moneoang ....5 - 263
Levenson, Esther ....
........................1 - 125, 3 - 307, 4 - 275
Li, Jian ....5 - 188
Li, Yeping ....1 - 187
Lilienthal, Achim J. ...... 1 - 209, 4 - 115
Liljedahl, Peter .... 1 - 189, 4 - 59
Lim, Dexter ....2 - 227
Lim, Woong ....4 - 147
Lima, Ewellen ....5 - 24
Lima, Izauriana Borges ....5 - 264
Lin, Fou-Lai ....5 - 189
Lin, Pi-Jen ....5 - 265
Lin, Su-Wei ....5 - 266
Lindmeier, Anke ....
..........................3 - 315, 4 - 379, 5 - 75
Lipowsky, Frank ....3 - 459
Lisarelli, Giulia .... 3 - 323, 5 - 212
Liu, Di ....5 - 104
Liu, Xin-Yi ....5 - 187, 5 - 194
Livy, Sharyn ....5 - 227
Llinares, Salvador ....
...........2 - 155, 2 - 187, 3 - 147, 5 - 237
Lo, Ya-Wen ....5 - 195
Lobanova, Anastasia ....5 - 185
López-Martín, María del Mar ....
.....................................2 - 131, 5 - 232
Losano, Ana Leticia ......3 - 331, 5 - 267
Lowenthal, Patrick ....5 - 213
Lu, Feng-Lin ....5 - 15, 5 - 300
Lu, Xiaoli ....5 - 156
Ludwig, Matthias ....5 - 52
Lundström, Marita ....5 - 105
Luxembourger, Christophe ....5 - 36

M

MacGregor, Stephen ....4 - 19
Machado, Ricardo ....5 - 225
Maciejewski, Wes ....3 - 339
Magiera, Marta ....5 - 79, 5 - 106
Maheux, Jean-François ....1 - 95
Mak, Soi-kei ...... 5 - 31, 5 - 268, 5 - 290
Makonye, Judah Paul ....3 - 403
Makramalla, Mariam Magdi Maurice...
....5 - 107
Malaspina, Martín ....5 - 108
Malaspina, Uldarico ......5 - 108, 5 - 176
Mali, Angeliki ....2 - 227, 5 - 109
Malmberg, Lars-Erik ....4 - 243
Mamona-Downs, Joanna ....5 - 110
Manizade, Agida ....1 - 197
Marmur, Ofer ....3 - 347
Martín-Molina, Verónica ....3 - 355
Martin, Lyndon ....1 - 155
Martínez Videla, María Victoria ....
......................................5 - 134, 5 - 286
Matos, João Filipe ....5 - 269
Matos, José Manuel ....5 - 221
McCallum, William G. ....1 - 213
McClintock, Evan ....5 - 78
McClure, Lynne ....1 - 213
McGarvey, Lynn ....1 - 155

McMullen, Jake ....3 - 363
Medrano, Ana ....4 - 451
Meehan, Maria............................3 - 371
Mégrourèche, Charlotte.............1 - 155
Melleone, Maria .... 1 - 247, 4 - 35
Merzel, Avi...............................4 - 139
Mesa, Vilma .... 2 - 227, 5 - 109
Mesiti, Carmel .... 2 - 435, 3 - 379
Meyer, Bronwin..........................5 - 183
Mgombelo, Joyce ....1 - 155
Micha, Ioanna ....3 - 395
Miragliotta, Elisa ....3 - 387
Misailidou, Christina ....5 - 299
Misfeldt, Morten.........................2 - 371
Miyakawa, Takeshi.....................5 - 157
Mizoguchi, Tatsuya ....5 - 157
Mizzi, Angel ....5 - 111
Mok, Ida Ah Chee .... 5 - 76, 5 - 310
Molina-Portillo, Elena ....5 - 232
Molina, Marta .... 5 - 16, 5 - 178
Montenegro, Helena ....5 - 149
Montoro Medina, Ana Belén......5 - 136
Montoya Delgadillo, Elizabeth...5 - 220
Moon, Sung-Jae ....5 - 112
Moreno Balcázar, Juan José ....5 - 136
Moreno, Mar................ 2 - 155, 4 - 107
Morita, Taisuke..........................5 - 258
Morselli, Francesca...............................
..........................1 - 85, 1 - 191, 2 - 323
Moschkovich, Judit........... 5 - 48, 5 - 71
Moutsios-Rentzos, Andreas........3 - 395
Movshovitz-Hadar, Nitsa ....5 - 270
Mudrikah, Achmad.....................5 - 113
Mueller, Henning.......................4 - 443
Murata, Shogo ....5 - 271
Mutara, Lydia ....3 - 403
Muzsnay, Anna..........................5 - 168
Mwakapenda, Willy ..... 5 - 115, 5 - 272

N

Naalsund, Margrethe ....5 - 114
Nachlieli, Talli ....5 - 63
Nagy, Caroline...........................5 - 287
Nardi, Elena..
...........4 - 363, 5 - 163, 5 - 172, 5 - 245
Narlı, Serkan .... 5 - 5, 5 - 44, 5 - 128
Nascimento, Marisa...................5 - 221
Ndlovu, Williams Chapasuka ....
....................................5 - 115, 5 - 272
Ndlovu, Zanele Annatoria.........5 - 116
Nemirovsky, Ricardo ....1 - 201
Neto, Vanessa...............1 - 203, 3 - 411
Neuhaus, Silke...........................5 - 117
Ng, Oi-Lam ....5 - 118
Ni Shuilleabhain, Aoibhinn........3 - 371
NicMhuirí, Siún.........................5 - 301
Nicol, Cynthia ....2 - 291
Niedermeyer, Inga.....................4 - 179
Nieminen, Juuso Henrik..4 - 299, 5 - 58
Nieminen, Pasi ....5 - 53
Nilsen, Trude ....3 - 179
Nilssen, Vivi..............................3 - 123
Niss, Mogens ....1 - 35
Nogues, Camila ....5 - 119, 5 - 273
Noh, Jihwa.................................5 - 120
Norén, Eva.................................5 - 304
Norqvist, Mathias ....
.... 3 - 419, 4 - 355, 5 - 302
Norton, Anderson ....4 - 323
Nortvedt, Guri ....
.... 1 - 205, 5 - 165, 5 - 274
Ntow, Forster.............................5 - 121
Núñez, Cristina..........................5 - 208
Nyman, Martin ....5 - 122
Nyman, Rimma ....5 - 123
Nyström, Peter...........................1 - 273

O

O'Brien, Kate.............................1 - 201
O'Brien, Katherine Claire...........5 - 275
Obersteiner, Andreas..................3 - 427
Oikkonen, Juha Markku Robert...........
....5 - 144
Oliveira, Hélia Margarida ....5 - 280
Olsher, Shai ....2 - 299
Olteanu, Constanta ....5 - 124

Olteanu, Lucian ....5 - 124
Onghena, Patrick ....5 - 166
Orrill, Chandra......................................
........................1 - 197, 5 - 125, 5 - 276
Ortiz, Andrea ....4 - 3
Ortiz, Andrés ....5 - 286
Oslington, Gabrielle Ruth...........3 - 435
Österholm, Magnus ....5 - 77
Österling, Lisa ....
........................1 - 211, 2 - 259, 3 - 443
Otaki, Koji ....5 - 241
Oteze, Kate Ikponmwosa ....5 - 230
Ouvrier-Buffet, Cecile.. 1 - 207, 5 - 126
Ovadiya, Tikva ....5 - 277
Özaltun Çelik, Aytuğ.... 5 - 127, 5 - 278

P

Pai, Yun Hsia.............................5 - 279
Pala, Ozan..................................5 - 128
Pan, Jiahui.................................5 - 129
Pandit, Ekaraj ....5 - 130
Pang, Ming Fai ....5 - 261
Papadopoulos, Ioannis.. 3 - 451, 5 - 238
Papageorgiou, Maria ....5 - 131
Partanen, Anna-Maija................5 - 132
Pásztor, Attila ....5 - 145
Paulding, Katie ....2 - 211
Pede, Stella ....3 - 459
Pedro, Ana ....5 - 269
Pedro, Neuza..............................5 - 269
Peña, Fredy ....4 - 3
Peng, Aihui ....5 - 133
Perdomo Díaz, Josefa ....
..........................5 - 61, 5 - 134, 5 - 286
Peteers, Florence.........................5 - 135
Pettersson, Kerstin.......................5 - 97
Pierri, Anna....................................5 - 7
Piñeiro, Juan Luis ....3 - 465
Pino-Fan, Luis R..........................5 - 25
Pinta, Maritza ....5 - 136
Pintér, Klára...............................5 - 169
Pinto, Alon.................................2 - 307
Pinto, Jorge ....5 - 137
Pinto, Márcia Maria Fusaro .... 1 - 69
Pitsili-Chatzi, Dionysia ....3 - 473
Pitt, Andrea ....2 - 51, 5 - 73
Pitta-Pantazi, Demetra ....2 - 235
Plaxco, David ....4 - 459
Podkhodova, Natalia ....5 - 138
Polo-Blanco, Irene......................5 - 280
Polotskaia, Elena ....
...........1 - 245, 1 - 263, 5 - 281, 5 - 288
Poncelet, Débora ....5 - 139, 5 - 174
Porter, Jessica ....3 - 51
Potari, Despina ....4 - 291
Poudel, Amrit Bahadur...............5 - 140
Poulopoulou, Stavroula..............2 - 147
Pratt, Nick ....2 - 11
Preciado Babb, Armando Paulino.. 4 - 3
Prediger, Susanne.............4 - 11, 4 - 99
Proulx, Jérôme............................1 - 155
Psycharis, Giorgos.........4 - 291, 4 - 395
Punzo, Colomba ....1 - 247
Pyper, Jamie S ....4 - 19

Q

Qiao, Xuefeng ....4 - 27, 5 - 282

R

Rach, Stefanie ....5 - 117, 5 - 141
Radišić, Jelena............................3 - 179
Ramdhany, Viren ....5 - 142
Ramirez, Paola ....5 - 143
Ramírez, Rafael............5 - 178, 5 - 284
Rämö, Johanna ....4 - 299, 5 - 58
Rangnes, Toril Eskeland ....4 - 227
Räsänen, Jenni Johanna..............5 - 144
Rathgeb-Schnierer, Elisabeth.....5 - 249
Rausch, Attila ....5 - 145
Redfors, Andreas ....5 - 243
Reinhold, Frank............................5 - 64
Reinhold, Simone ....5 - 227
Reis, Maria ....1 - 125
Reiss, Kristina ... 4 - 251, 5 - 64, 5 - 102
Reitz-Koncebovski, Karen ....5 - 283
Rekhtman, Iya ....5 - 185

Remillard, Janine........................3 - 259
Resnick, Lauren B. ....3 - 363
Reyes Santander, Pamela .... 5 - 146, 5 - 220
Reyes, Cristian...........................5 - 286
Rhoads, Kathryn........................5 - 147
Ribeiro, Miguel............................4 - 35
Ribera, Juan Miguel...................5 - 284
Richter-Gebert, Jürgen ....5 - 64
Riesten, Laila.............................5 - 148
Rietenberg, Anja........................4 - 379
Rim, Haekyung...........................5 - 257
Rizza, Davide ....3 - 131
Robotti, Elisabetta .... 1 - 207, 4 - 43
Rodd, Melissa............................2 - 315
Rodrigues, Margarida Maria ......5 - 155
Rodrigues, Maria Paula Pereira............ ....5 - 285
Rodríguez, Dolores....................5 - 208
Rogovchenko, Svitlana..............4 - 283
Rogovchenko, Yuriy ....4 - 283
Röj-Lindberg, Ann-Sofi ....4 - 203, 5 - 84, 5 - 150
Rojas, Francisco ....5 - 134, 5 - 149, 5 - 286
Rolka, Katrin ....2 - 451
Roorda, Gerrit...................................... ....4 - 387, 5 - 3, 5 - 27, 5 - 151
Rotem, Sigal ....4 - 51
Rott, Benjamin..................................... ....3 - 171, 4 - 59, 5 - 153, 5 - 294
Rouleau, Annette .... 1 - 189, 4 - 67
Runesson Kempe, Ulla ....1 - 125, 3 - 267, 5 - 42
Ruttenberg-Rozen, Robyn .... 1 - 215, 5 - 152
Ruwisch, Silke.............................3 - 67
Ryan, Ulrika ....4 - 75
Ryoo, Byeongguk........................4 - 83

S

Saas, Hannes................................5 - 75
Sabena, Cristina........... 1 - 191, 1 - 245
Saenz-Ludlow, Adalira ....4 - 91
Safrudiannur, Safrudiannur........5 - 153
Säfström, Anna Ida.....................5 - 287
Sahin-Gür, Dilan ....4 - 11, 4 - 99
Sakonidis, Charalampos....................... ....2 - 115, 4 - 475
Salameh, Rania...........................4 - 163
Salgado, María ....5 - 208
Salinas, María Jesús ....5 - 208
Salle, Alexander ....3 - 27
Sanchez-Matamoros, Gloria.......4 - 107
Sánchez, Alicia...............5 - 25, 5 - 154
Sandoval Rosas, Marisol................4 - 3
Santiago, Ana ....5 - 225
Santos, Leonor..................................... .... 1 - 205, 2 - 203, 5 - 137
Sarama, Julie ....1 - 125
Savard, Annie ....1 - 263, 5 - 214, 5 - 281, 5 - 288
Schäfer, Ingolf..............................5 - 20
Schindler, Maike .... 1 - 209, 3 - 171, 4 - 115
Schukajlow, Stanislaw .... 1 - 195, 2 - 163, 3 - 243
Schumacher, Stefanie...................3 - 27
Schunk, Daniel ....4 - 443
Schwarts, Gil ....4 - 123
Seah, Rebecca ....4 - 131
Seah, Wee Tiong ....1 - 85
Sealey, Vicki Lynn.....................2 - 171
Segal, Ruti .... 4 - 139, 5 - 270, 5 - 277
Selva, Ana Coelho Vieira...........5 - 264
Semanišinová, Ingrid....................5 - 70
Seo, Eun-Min ....4 - 147
Sequeiros, Pablo G. ....5 - 208
Serrazina, Lurdes...........5 - 155, 5 - 285
Sezgin, H. Seda ....5 - 4, 5 - 289
Shabtay, Galit.............................4 - 155
Shahbari, Juhaina Awawdeh ......4 - 163
Shang, Yueqiang ....5 - 133
Shi, Jie ....5 - 156
Shin, Bomi..................................5 - 257
Shinno, Yusuke ....5 - 157

Shriki, Atara ....5 - 270
Shults, F. LeRon ....5 - 140
Shvarts, Anna .... 1 - 209, 4 - 171
Sievert, Henning ....4 - 179
Silva, Marcio ....4 - 187
Simmt, Elaine ....1 - 155
Simon, Martin ....4 - 195
Sinclair, Nathalie ....
...............1 - 51, 1 - 201, 2 - 99, 2 - 275
Singletary, Laura Marie.............2 - 283
Sit, Pou Seong ...5 - 31, 5 - 268, 5 - 290
Sjöblom, Marie ....5 - 291
Skidras, Christina ....5 - 123
Skilling, Karen ....5 - 158
Skog, Kicki ....1 - 211
Smedlund, Joakim Vilhelm ....4 - 203
Smith, Amy ....4 - 323
Smith, Cathy Anne ....4 - 211
Sodian, Beate ....2 - 331
Solares Rojas, Armando ....4 - 3
Sommerhoff, Daniel ..... 5 - 159, 5 - 292
Son, Ji-Won ....3 - 211
Song, Shuang ....4 - 219
Soon, Tan Liang ....5 - 103
Soriano Velasco, Remedios ....4 - 3
Sproesser, Ute ....5 - 160
Staats, Susan Kimberley ....
.... 5 - 161, 5 - 293
Steffensen, Lisa ....4 - 227
Steiner, Aya ....4 - 235
Steinthorsdottir, Olof Bjorg........5 - 162
Stenning, Keith ....5 - 181
Stoppel, Hannes ....5 - 294
Street, Karin Elisabeth Sørlie .....4 - 243
Strohmaier, Anselm R. ....4 - 251
Strom, April ....2 - 227
Stylianides, Andreas ..... 1 - 213, 5 - 107
Stylianides, Gabriel J. ....4 - 243
Stylianidou, Angeliki ....5 - 163
Subramanian, Jayasree ....5 - 179
Subtil, Manuela ....5 - 164
Sue, Naoki ....3 - 227
Sugimoto, Shin ....5 - 258
Suhre, Cor ....5 - 27
Sumpter, Lovisa ....5 - 165, 5 - 274
Sun, Sijie ....5 - 156
Sun, Wei-Min ....5 - 298
Supply, Anne-Sophie ....5 - 166
Suriakumaran, Neruja ....5 - 167
Swanson, Steve ....5 - 305
Swidan, Osama ....4 - 259, 5 - 54
Szabó, Csaba ... 5 - 168, 5 - 295, 5 - 296
Szeibert, Janka ....5 - 296
Szitányi, Judit ....5 - 9, 5 - 169

T

Tabach, Michal ....2 - 27
Taghi Dastjerdi, Sharareh ....2 - 67
Takenoya, Syuuta ....5 - 258
Takeuchi, Ayumu ....5 - 297
Tam, Hak Ping ....5 - 298
Tang, Jinbo ....5 - 76
Tanışlı, Dilek ....5 - 170, 5 - 196
Tasara, Innocent ....4 - 267
Taylan, Rukiye Didem ....5 - 171
Tazouti, Youssef ....5 - 139, 5 - 174
Teledahl, Anna ....1 - 205
Thanheiser, Eva ....5 - 94
Thibault, Mathieu ....2 - 107
Thom, Jennifer ....1 - 155
Thoma, Athina ....3 - 131, 5 - 172
Thomas, Mike ....5 - 46
Thouless, Helen Rachel ....
.....................................1 - 215, 5 - 173
Tièche Christinat, Chantal..........5 - 139
Tinnes-Vigne, Mélanie ....
.... 5 - 36, 5 - 139, 5 - 174
Tirosh, Dina ....1 - 125, 4 - 275
Toh, Tin-Lam ....5 - 81
Toh, Wei-Yeng Karen ....5 - 81
Toivanen, Miika ....2 - 475
Tolvanen, Pieti ....5 - 132
Torbeyns, Joke ....5 - 175
Torres Ninahuanca, Carlos ....5 - 176
Toscano, Rocío ....3 - 355
Totorica, Tatia ....5 - 213

Towers, Jo....................................1 - 155
Treffert-Thomas, Stephanie..................
.... 4 - 283, 5 - 62
Triantafillou, Chrissavgi.......................
.... 4 - 291, 4 - 395
Tsai, Bao-Quei............................5 - 265
Tsamir, Pessia.............. 1 - 125, 4 - 275
Tsikalaki, Eleni...........................5 - 299
Tsiplaki, Ioanna..........................2 - 147
Tso, Tai-Yih .... 5 - 15, 5 - 300
Tu, Shu-Min ....5 - 69
Tun, Sena Simay.........................3 - 203
Tuohilampi, Laura........ 1 - 235, 4 - 299
Tuomela, Tatu Dimitri................4 - 307
Turgut, Melih.....4 - 315, 5 - 26, 5 - 170
Turineck, Louis-Philippe............5 - 214
Twohill, Aisling..................................
.........................1 - 213, 5 - 177, 5 - 301
Tzekaki, Marianna......................5 - 131
Tzur, Ron.........1 - 215, 3 - 163, 4 - 323

U

Uegatani, Yusuke ....3 - 43
Ufer, Stefan.................. 2 - 331, 5 - 159
Ulloa, Rodrigo............. 5 - 134, 5 - 286
Ureña, Jason ....5 - 178
Urrutia, Mabel .... 5 - 50, 5 - 236
Ursekar, Chaitanya Prashant ......5 - 179
Uygan, Candas............... 5 - 26, 5 - 180

V

Vale, Pamela...............................4 - 331
Valero, Paola ....
.............1 - 95, 3 - 411, 4 - 187, 5 - 304
Valls, Julia ....4 - 107
Vamvakoussi, Xenia...................2 - 147
Van Den Ham, Ann-Kathrin ......4 - 179
Van Dooren, Wim .... 1 - 77, 1 - 195, 2 - 459, 3 - 299, 4 - 347, 5 - 166
Van Hoof, Jo..............................4 - 339
van Steenbrugge, Hendrik ....
.... 3 - 259, 5 - 302
Vanegas, Yuly ....5 - 303
Vanluydt, Elien ....4 - 347
Varga, Eszter ....5 - 9
Vargas, Francisco Javier ....5 - 181
Venenciano, Linda C.................5 - 182
Venkat, Hamsa .... 1 - 125, 2 - 75, 5 - 57
Vennberg, Helena.......................4 - 355
Verguts, Gwen............................5 - 175
Vermeulen, Cornelis ....5 - 183
Verschaffel, Lieven ....
.............4 - 339, 4 - 347, 5 - 37, 5 - 175
Viiri, Jouni...................................5 - 53
Viirman, Olov ....4 - 363
Viitala, Hanna ....5 - 184
Vivier, Laurent ....5 - 220
Vlaseros, Nikolaos ....5 - 62
Vlassis, Joelle.... 4 - 371, 5 - 36, 5 - 174
Vogel, Markus.............................5 - 160
Vollstedt, Maike.........................5 - 167
vom Hofe, Rudolf.......................5 - 146
von Hering, Robert.....................4 - 379
Vos, Pauline ....4 - 387, 5 - 140
Vroutsis, Nikolaos......................4 - 395
Vysotskaya, Elena......................5 - 185

W

Wagner, David ....5 - 186
Waisman, Ilana...........................4 - 403
Wallin, Anna ....5 - 304
Wan, May Ee Vivien.................1 - 225
Wang, Chuang............................5 - 254
Wang, Fang-Jiun ....5 - 187
Wang, Guangming ....5 - 188
Wang, Sasha ....5 - 305
Wang, Ting-Ying ....5 - 67, 5 - 189
Wang, Xin ....3 - 163
Watanabe, Satoshi ....5 - 306
Watkins, Laura ....2 - 227
Wedman, Lotta ....4 - 411
Wei, Bingqian ....3 - 163, 4 - 323
Weigand, Hans-Georg................4 - 419
Weiher, Dana Farina ....3 - 67, 5 - 307
Weiland, Travis ....5 - 125
Weingarden, Merav.........4 - 427, 5 - 63

Werner, Bernhard ....5 - 64
Wessman-Enzinger, Nicole..........5 - 23
Westaway, Lise...........................5 - 190
Weston, Tracy L. ....5 - 191
Whipple, Kyle.............................5 - 161
Wickstrom, Megan H. ....5 - 192
Widder, Mirela ....4 - 435
Wilkie, Karina ....2 - 91
Williams-Pierce, Caro ....4 - 459
Williams, Gaye ....5 - 193
Willoughby, Laura........................5 - 65
Winkel, Kirsten...........................4 - 443
Wöller, Susanne..........................5 - 227
Wong, Ka Lok ....5 - 261
Wu, Huei-min............... 5 - 253, 5 - 308
Wu, Lan-Ting ....5 - 189
Wu, Libao ....5 - 188

X

Xolocotzin, Ulises ....4 - 451

Y

Yagi, Seanyelle L. ....5 - 182
Yamazaki, Miho ....3 - 227
Yang, Dazhi ....5 - 305
Yang, Kai-Lin ....
............5 - 187, 5 - 189, 5 - 194, 5 - 195
Yanishevskaya, Maria ....5 - 185
Yazgan, Yeliz ....5 - 90
Yıldırım, Duygu ....5 - 196
Yılmaz, Süha ....5 - 38
Yu, Bo ....5 - 133

Z

Zachariades, Theodosios ....
....2 - 115, 4 - 475
Zambak, Vecihi S.......................5 - 106
Zandieh, Michelle ....4 - 459
Zeynivandnezhad, Fereshteh................
....5 - 197, 5 - 309
Zhang, Nan ....5 - 188
Zhang, Qiaoping.............1 - 77, 5 - 198
Zhang, Shu ....5 - 211
Zhao, Wenjun ....5 - 310
Zhu, Yan ....5 - 156
Zindel, Carina.................4 - 11, 4 - 467
Zlatkin-Troitschanskaia, Olga......5 - 75
Zoitsakos, Sotirios......................4 - 475

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